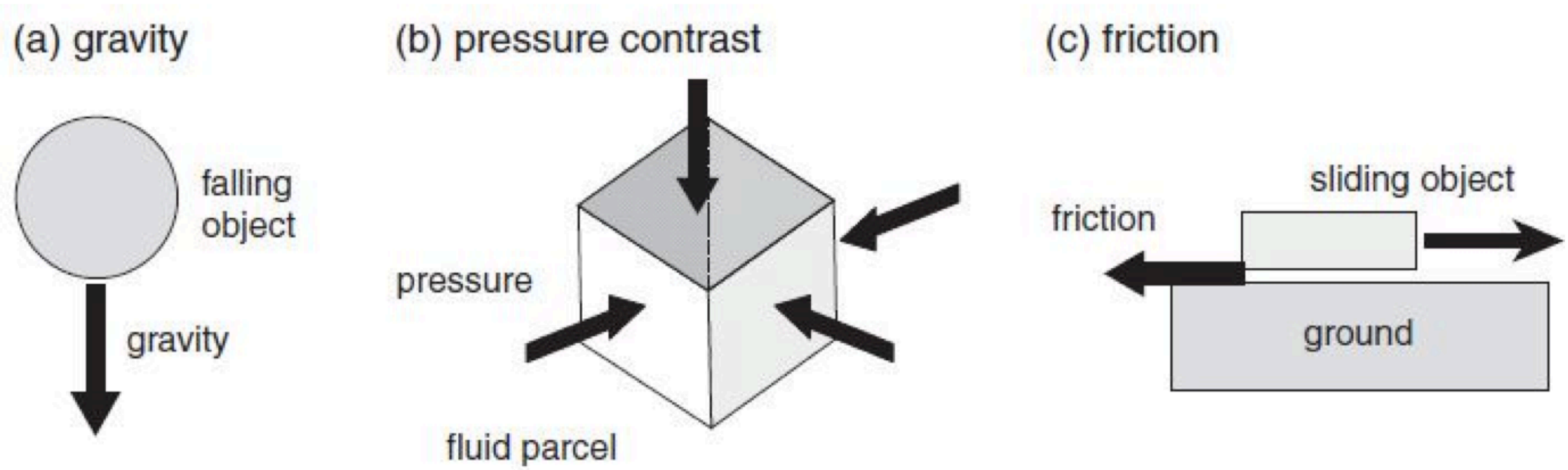
Week 4: Fluid dynamics principles

Tally’s book, chapter 7

Balance of forces

Three types of forces we consider here:



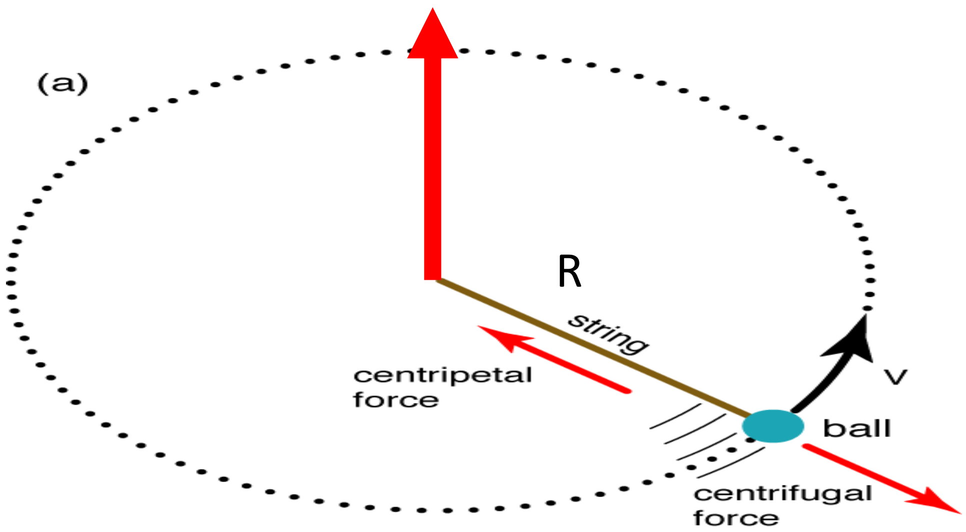
And the effect of the rotation of the Earth

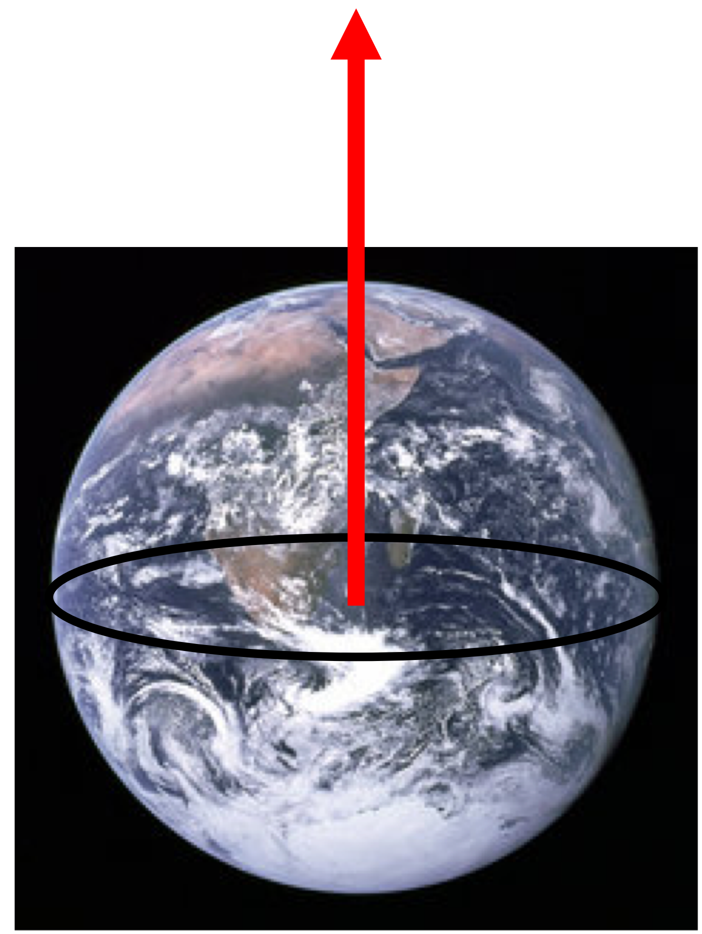
- **Centrifugal force**
- **Coriolis force**

Rotating coordinates

- **Rotation vector** that expresses direction of rotation and how fast it is rotating:

  vector pointing in direction of thumb using right-hand rule, curling fingers in direction of rotation





Ω = angle/second ~ 10^{-4} (s^{-1})

Frame of reference: mathematical expression

- Inertial frame of reference (non-rotating) : subscript "in"
  – Velocity in inertial frame: $\mathbf{u}_{in}$

- Rotating frame of reference : subscript "rot"
  – Velocity in rotating frame: $\mathbf{u}_{in} = \mathbf{u}_{rot}$ + (effect of rotation)

$$\mathbf{u}_{in} = \mathbf{u}_{rot} + \Omega \times \mathbf{x}$$

$$\left(\frac{d\mathbf{x}}{dt}\right)_{in} = \left(\frac{d\mathbf{x}}{dt}\right)_{rot} + \Omega \times \mathbf{x}$$

Parcel acceleration in the rotating frame

- Apply the formula twice

$$\left(\frac{d\mathbf{u}_{in}}{dt}\right)_{in} = \left(\frac{d\mathbf{u}_{in}}{dt}\right)_{rot} + \Omega \times \mathbf{u}_{in}$$

$$= \left(\frac{d\mathbf{u}_{rot}}{dt}\right)_{rot} + \underbrace{2\Omega \times \mathbf{u}_{rot}}_{Coriolis} + \underbrace{\Omega \times \Omega \times \mathbf{x}}_{Centrifugal}$$

- Two forces
  - Coriolis force depends on **parcel velocity**
  - Centrifugal force depends on **parcel position**

Equation of motion in the rotating frame

- Expression of the Centrifugal term

$$-\Omega \times \Omega \times \mathbf{x} = \Omega^2 r_0 = \frac{\partial}{\partial r_0}\left(\frac{1}{2}\Omega^2 r_0^2\right) = -\nabla \Phi_{ce}$$

where $\Phi_{ce} = -\frac{1}{2}\Omega^2 r_0^2$

  - Radially outward ($+r_0$) w.r.t. the axis of rotation
  - Φ_{ce} is called centrifugal potential

Local Cartesian coordinate

- Local Cartesian (x,y,z), ignoring sphericity
- West-east (“zonal”) = x – direction
  - positive eastwards
- South-north (“meridional”) = y-direction
  - positive northwards
- Down-up (“vertical”) = z-direction
  - positive upwards

Effective gravity

Can we combine gravitational and centrifugal force?

- Both of them are characterized by potentials (Φ), thus can be combined together

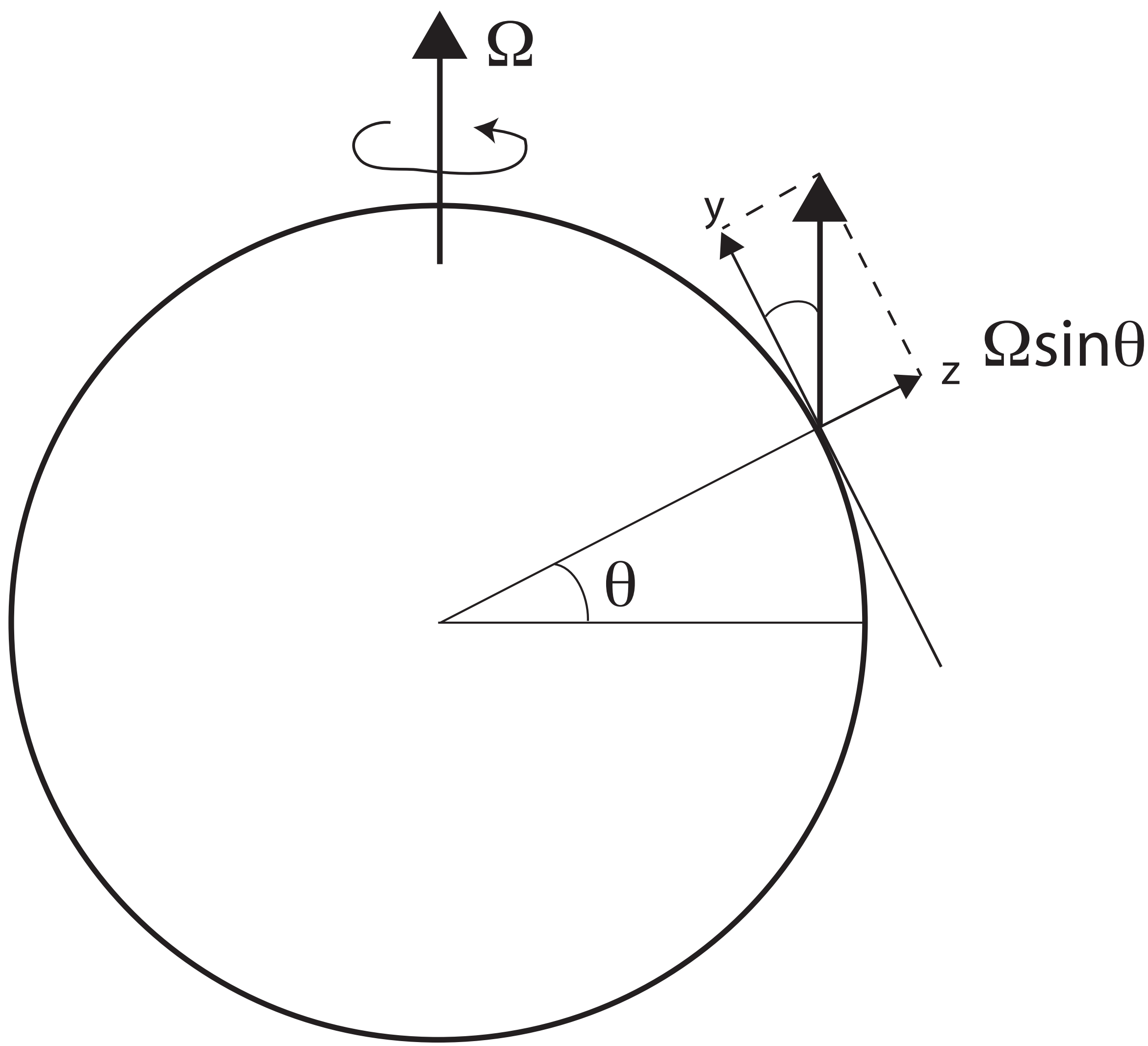
$$\frac{D\mathbf{u}}{Dt} + 2\Omega \times \mathbf{u} = \underbrace{-g\mathbf{k} + \Omega^2 \mathbf{r_0}}_{\text{Effective gravity}} = -\nabla\Phi = -g^*\mathbf{k}$$

$$\Phi = gz - \frac{1}{2}\Omega^2 r_0^2$$

Combined gravitational and centrifugal potential

Rotation vector in local Cartesian coordinate

- The rotation vector projects onto meridional (y, north-south) and vertical (z) directions
- Vertical coordinate is defined in the direction of effective gravity



Coriolis force

In local Cartesian coordinate, the Coriolis force depends on the horizontal speed (u,v) and the Coriolis parameter ($f=2\Omega\sin\theta$).

$$2\boldsymbol{\Omega}\times\mathbf{u} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ 0 & 2\Omega\cos\theta & 2\Omega\sin\theta \\ u & v & w \end{vmatrix}$$

$$= (2\Omega w\cos\theta - 2\Omega v\sin\theta)\mathbf{x} + 2\Omega u\sin\theta\mathbf{y} - 2\Omega u\cos\theta\mathbf{z}$$

$$\approx -2\Omega v\sin\theta\mathbf{x} + 2\Omega u\sin\theta\mathbf{y} - 2\Omega u\cos\theta\mathbf{z}$$

$$\approx -fv\mathbf{x} + fu\mathbf{y}$$

Equation of motion in local Cartesian coordinate

In x-, y-, z- components

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{F_x}{\rho}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{F_y}{\rho}$$

$$\frac{Dw}{Dt} = -g - \frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{F_z}{\rho}$$

Equation of motion in local Cartesian coordinate

Coriolis force

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{F_x}{\rho}$$

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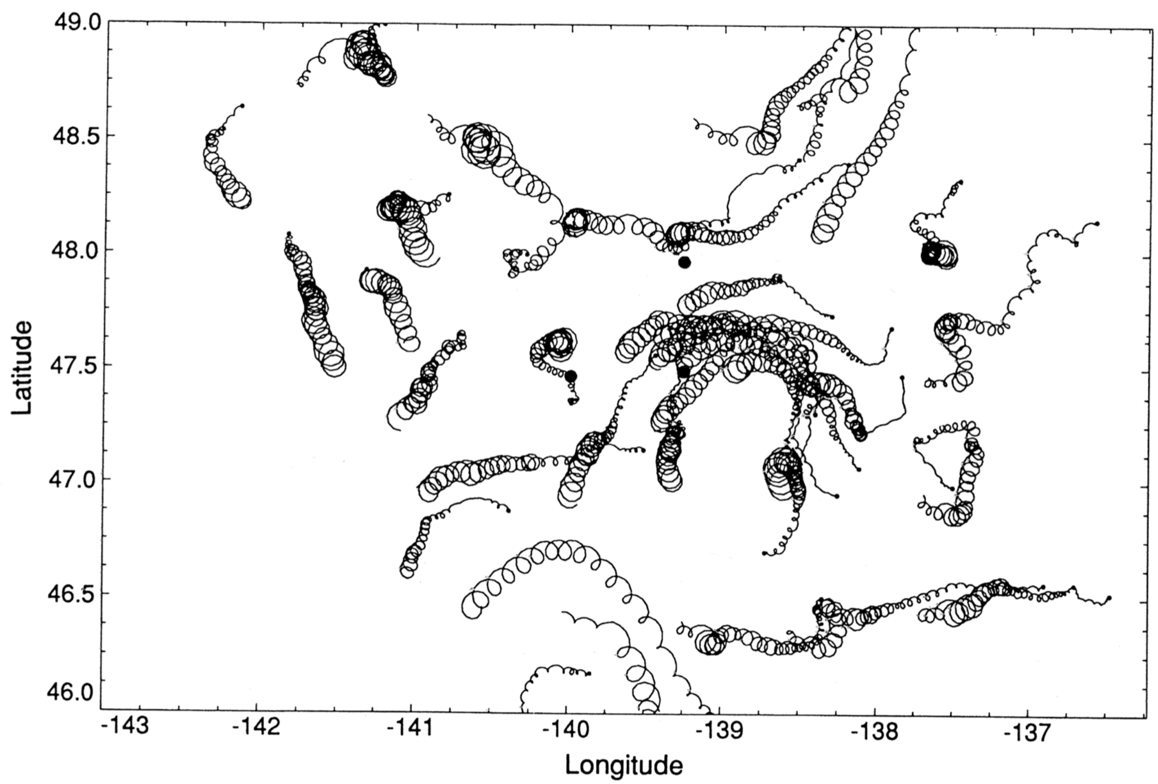
$$\frac{Dw}{Dt} = -g - \frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{F_z}{\rho}$$

Coriolis in action in the ocean: Observations of Inertial Currents



D'Asaro et al. (1995)

DPO Fig. 7.4

- Surface drifters in the Gulf of Alaska during and after a storm.
- Note the corkscrews - drifters start off with clockwise motion, which gets weaker as the motion is damped (friction)

Inertial oscillation

Balance between local acceleration and Coriolis force

$$\frac{D\mathbf{u}}{Dt} + 2\Omega \times \mathbf{u} = 0 \quad \longrightarrow \quad \begin{aligned} \frac{\partial u}{\partial t} &= fv \\ \frac{\partial v}{\partial t} &= -fu \end{aligned}$$

Solving this equation results in oscillatory solution with the Coriolis frequency ➔ Inertial oscillation

Equation of motion in local Cartesian coordinate

Pressure gradient force

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{F_x}{\rho}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{F_y}{\rho}$$

$$\frac{Dw}{Dt} = -g - \frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{F_z}{\rho}$$

Equation of motion in local Cartesian coordinate

Effective gravity (gravity + centrifugal force)

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{F_x}{\rho}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{F_y}{\rho}$$

$$\frac{Dw}{Dt} = \boxed{-g} - \frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{F_z}{\rho}$$

Equation of motion in local Cartesian coordinate

Hydrostatic balance (gravity = pressure force)

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{F_x}{\rho}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{F_y}{\rho}$$

$$\frac{Dw}{Dt} = \boxed{-g - \frac{1}{\rho}\frac{\partial P}{\partial z}} + \frac{F_z}{\rho}$$

Primitive equation

- **Equation of motion:** 3 coupled partial differential equations for (u,v,w) with 5 unknows (u, v, w, P, ρ)
  - More equations are needed to determine the solution
- **Conservation of mass** (continuity equation)
- **Equation of state** $\rho = \rho(S,T,P)$
  - Density depends on salinity and temperature
  - **Conservation of heat** (pot. Temperature)
  - **Conservation of salinity**
- Total of 6 coupled non-linear PDEs and the equation of state: very difficult to solve and understand, so **we attempt to derive simplified equations for approximate solutions**

Geostrophic balance

Geostrophic balance : Coriolis and Pressure gradient

X (east-west)

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{F_x}{\rho}$$

Y (north-south)

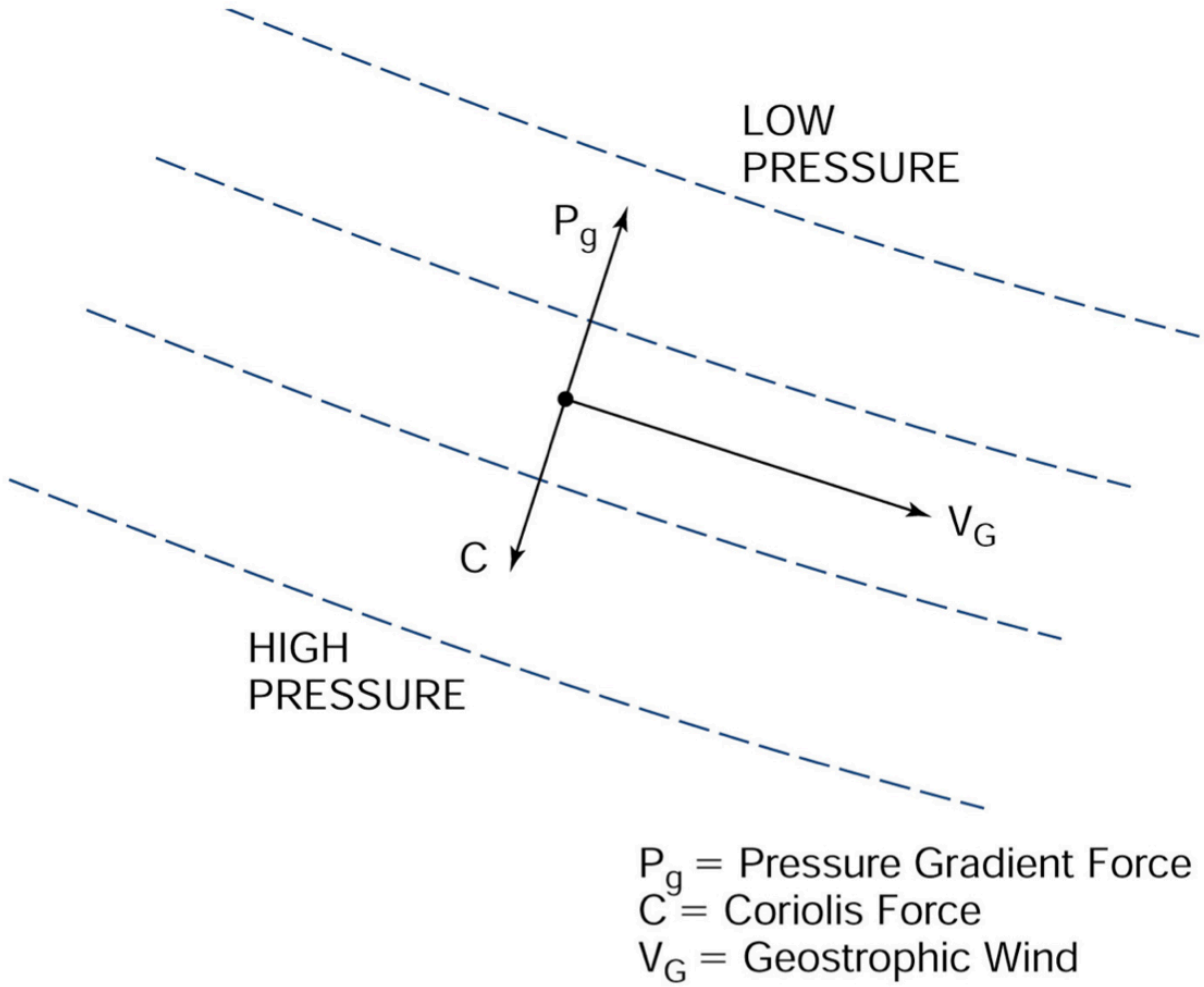
$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{F_y}{\rho}$$

Z (vertical)

$$0 = -g - \frac{1}{\rho}\frac{\partial P}{\partial z}$$

Geostrophy: PGF balanced by Coriolis force

Northern hemisphere: flow to the right of the PGF



What controls the pressure gradient force?

Hydrostatic pressure

$$\frac{\partial P}{\partial z} = -\rho g$$

$$P(z) = +\int_{z}^{\eta} \rho g \, dz + p_S$$

What controls the pressure gradient force?

Hydrostatic pressure

$$\frac{\partial P}{\partial z} = -\rho g$$

Sea surface height can change in x and y

$$P(z) = +\int_{z}^{\eta} \rho g \, dz + p_s$$

What controls the **pressure gradient** force?

Hydrostatic pressure

$$\frac{\partial P}{\partial z} = -\rho g$$

$$P(z) = +\int_{z}^{\eta} \rho g \, dz + p_S$$

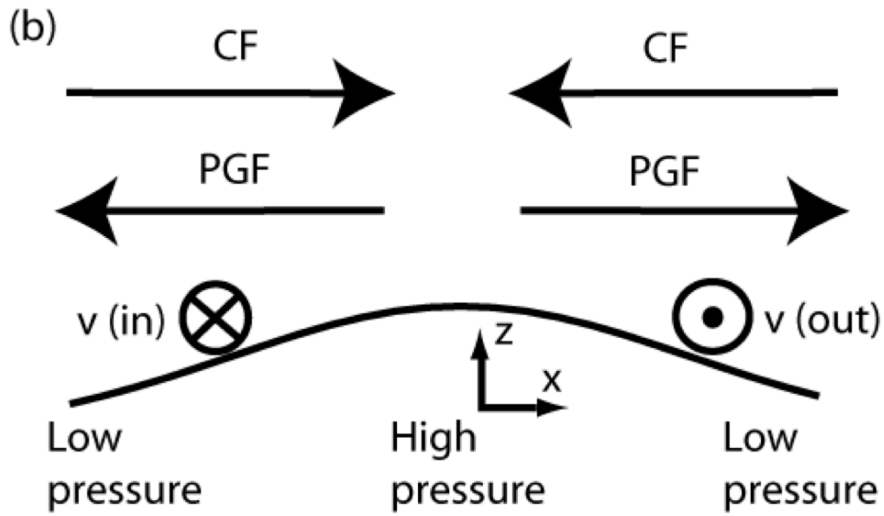
Sea surface height can change in x and y

Density can change in x and y (and z)

Observing the ocean's geostrophic current

- What we want: **current speed and direction**
- Required observation: **pressure gradient**
  - Before 1990s, it was impossible to determine the spatial distribution of sea surface height accurate enough for the geostrophic calculation
  - Density (T, S, P) measurements were available and used to estimate geostrophic currents
  - Satellite altimeter (since 1990s) can determine sea level height with an accuracy within a few cm

Pressure gradient due to SSH gradient



$$\frac{\partial P}{\partial z} = -\rho_0 g$$

$$P(x, y, z) = p_S - \rho_0 g \{z - \eta(x, y)\}$$

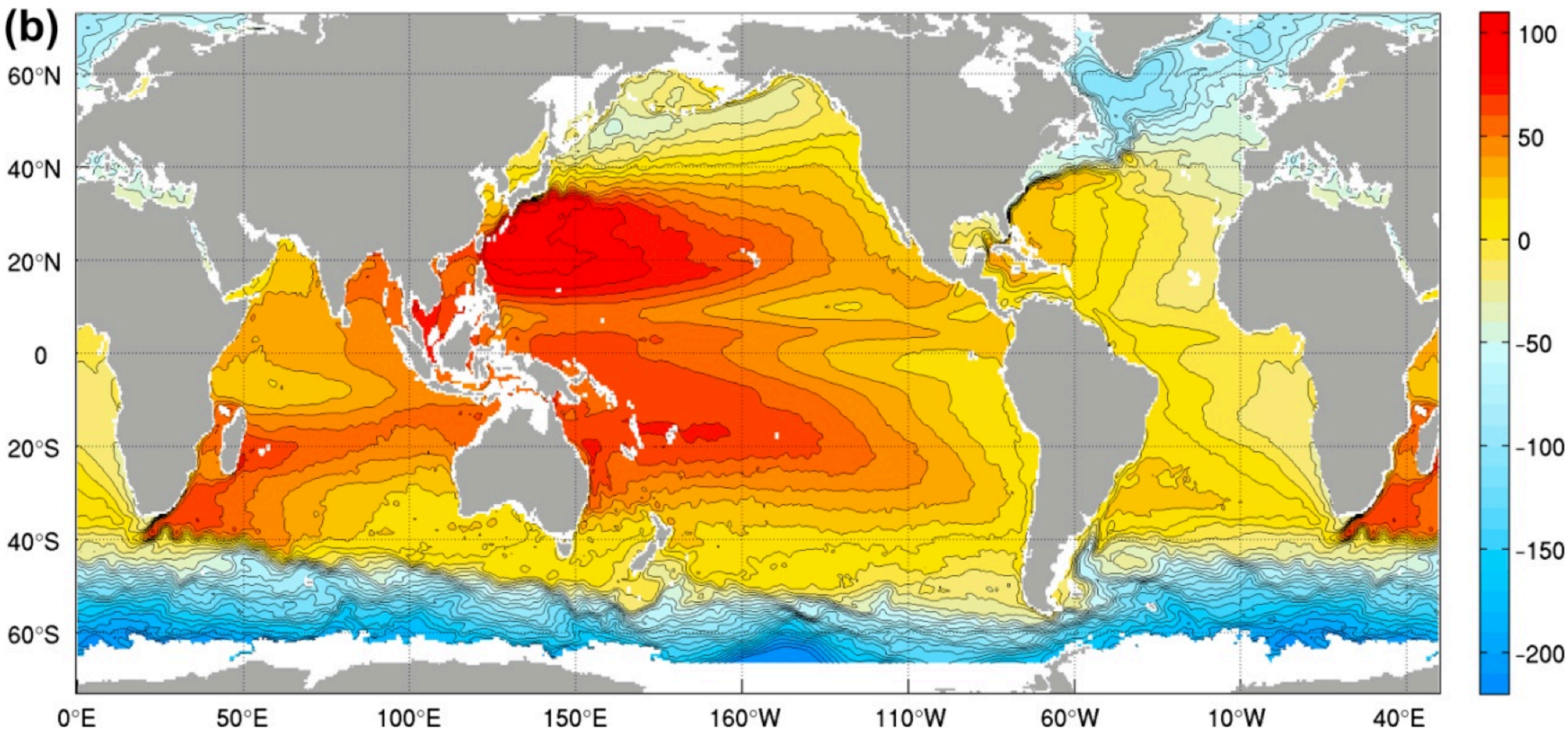
ps : a constant surface atmospheric pressure (Nm^{-2})

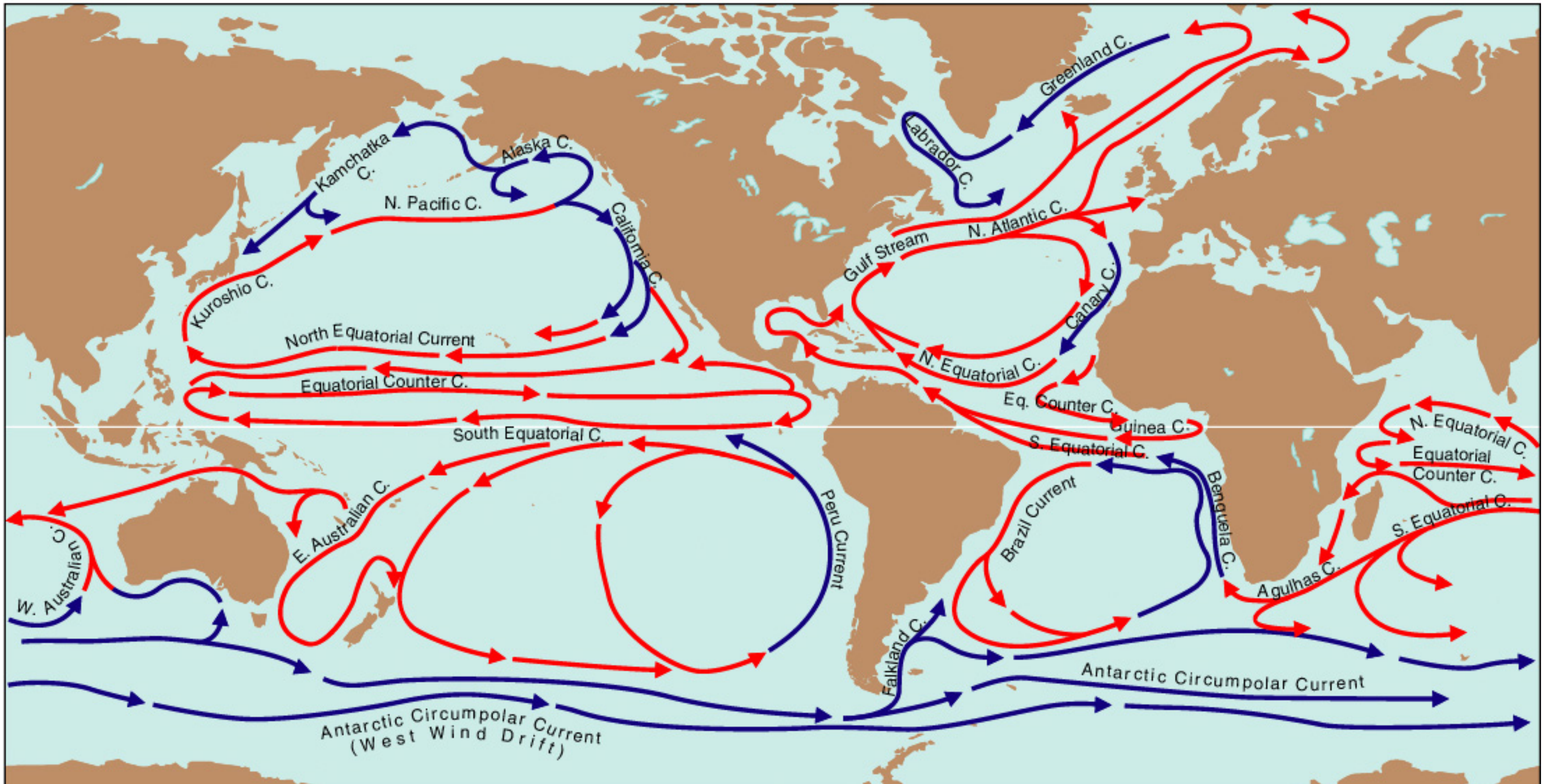
η: sea surface height or SSH (m)

$$\frac{\partial P}{\partial x} = \rho_0 g \frac{\partial \eta}{\partial x}, \quad \frac{\partial P}{\partial y} = \rho_0 g \frac{\partial \eta}{\partial y}$$

Sea surface height

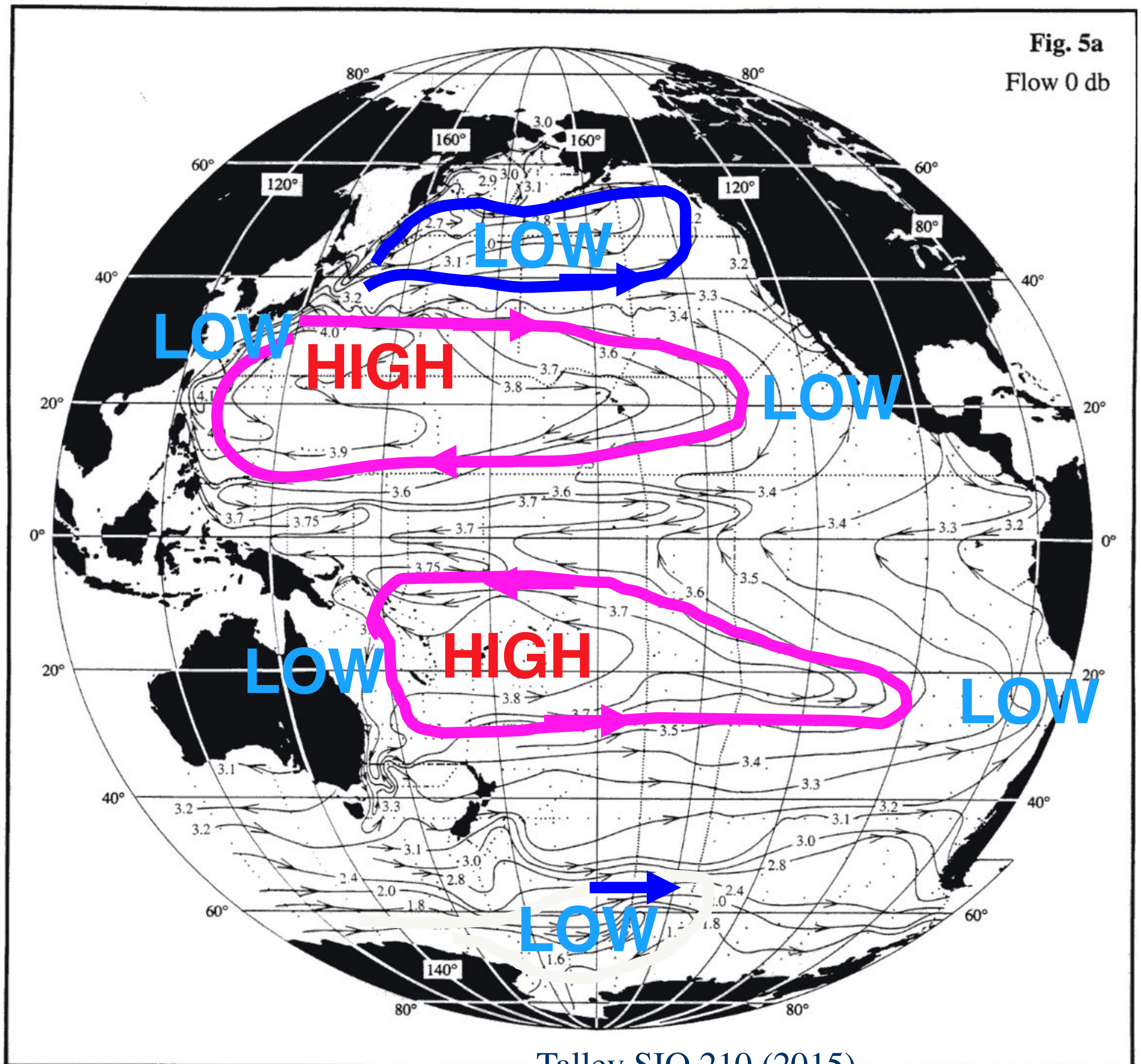
Compare with surface circulation (next slide)





Warm-water current   Cold-water current

Example from ocean of geostrophic flow
Surface height/pressure and surface geostrophic circulation



Talley SIO 210 (2015)

Circulation is counterclockwise around the Low (cyclonic) and clockwise around the High (anticyclonic).

(northern hemisphere)

Southern Hem:

Cyclonic is clockwise, etc..

Reid, 1997

Geostrophic approximation

- Under what condition can we assume geostrophic balance?
- The equation of motion: estimate the magnitude of each term

$$\frac{\partial u}{\partial t} \qquad + \mathbf{u} \cdot \nabla u \qquad - fu = \qquad -\frac{1}{\rho}\frac{\partial P}{\partial x} \qquad + \frac{1}{\rho}F_x$$

- U: velocity scale, T: time scale, L: length scale, ρ: density scale, P: pressure scale, F: friction scale

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad fU \qquad \frac{P}{\rho L} \qquad \frac{F}{\rho}$$

Geostrophic approximation

- Estimate the magnitude of each term
- Non-dimensionalize : normalize every term by fU

$$\frac{\partial u}{\partial t} \qquad + \mathbf{u}\cdot\nabla u \qquad - fu = \qquad -\frac{1}{\rho}\frac{\partial P}{\partial x} \qquad + \frac{1}{\rho}F_x$$

$$\frac{U}{T} \qquad \frac{U^2}{L} \qquad fU \qquad \frac{P}{\rho\, L} \qquad \frac{F}{\rho}$$

$$\frac{1}{fT} \qquad \frac{U}{fL} \qquad 1 \qquad \frac{P}{\rho\, f\, L\, U} \qquad \frac{F}{\rho\, f\, U}$$

Geostrophic approximation

- Estimate the magnitude of each term
- Non-dimensionalize : normalize every term by fU

$$\frac{\partial u}{\partial t} \qquad + \mathbf{u} \cdot \nabla u \qquad - fu = \qquad -\frac{1}{\rho}\frac{\partial P}{\partial x} \qquad + \frac{1}{\rho} F_x$$

$$\frac{1}{fT} \qquad \frac{U}{fL} \qquad \boxed{1 \qquad \frac{P}{\rho f L U}} \qquad \frac{F}{\rho f U}$$

Geostrophic approximation assumes **slow, large-scale circulation** and **weak friction**

$$\frac{1}{fT}, \ \frac{U}{fL}, \ \frac{F}{\rho f U} \ll 1$$

Expression of geostrophic flow using SSH

$$u_g = -\frac{1}{\rho f}\frac{\partial P}{\partial y},$$

$$v_g = \frac{1}{\rho f}\frac{\partial P}{\partial x},$$

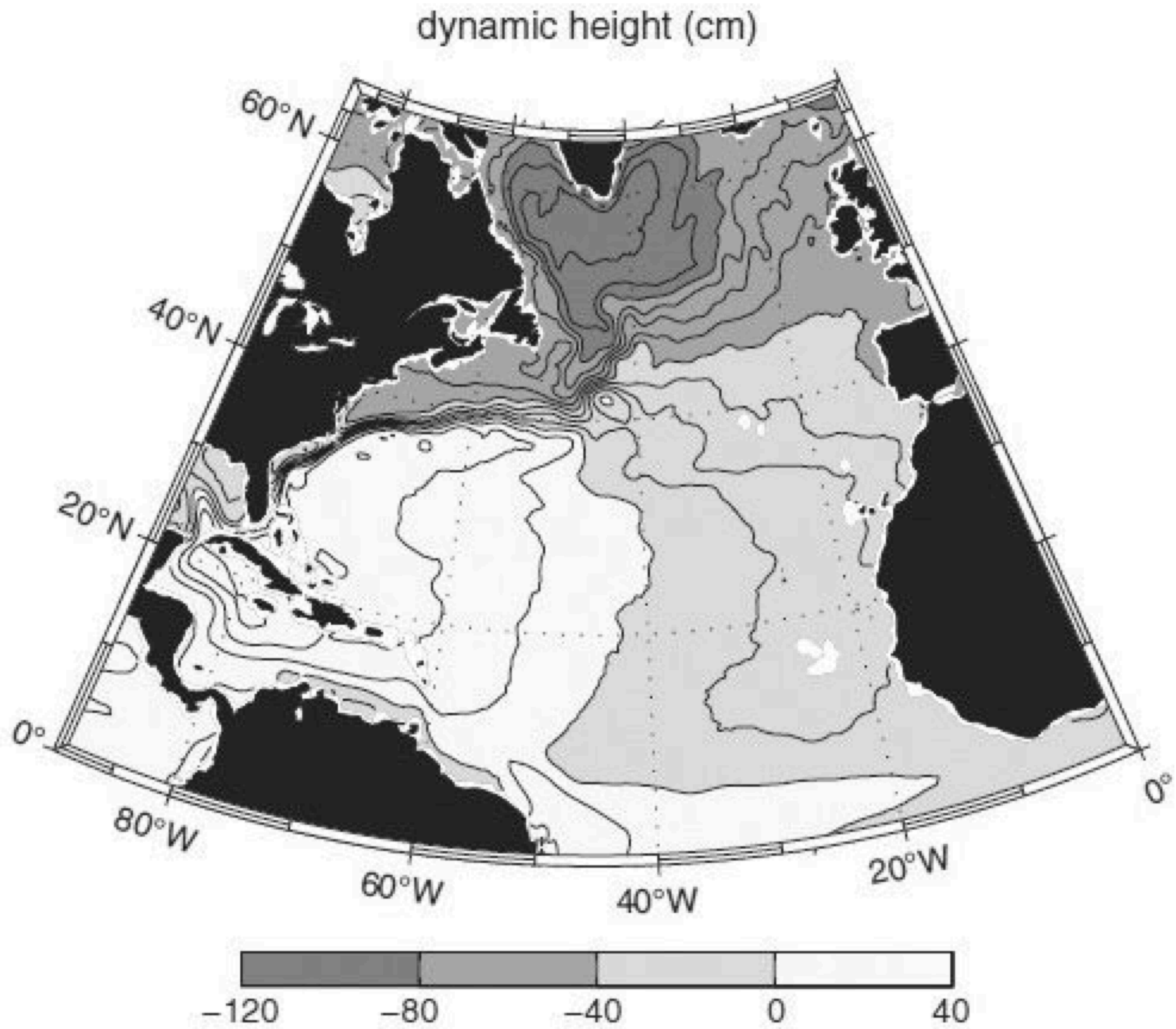
$\longrightarrow$

$$u_g = -\frac{g}{f}\frac{\partial \eta}{\partial y},$$

$$v_g = \frac{g}{f}\frac{\partial \eta}{\partial x},$$

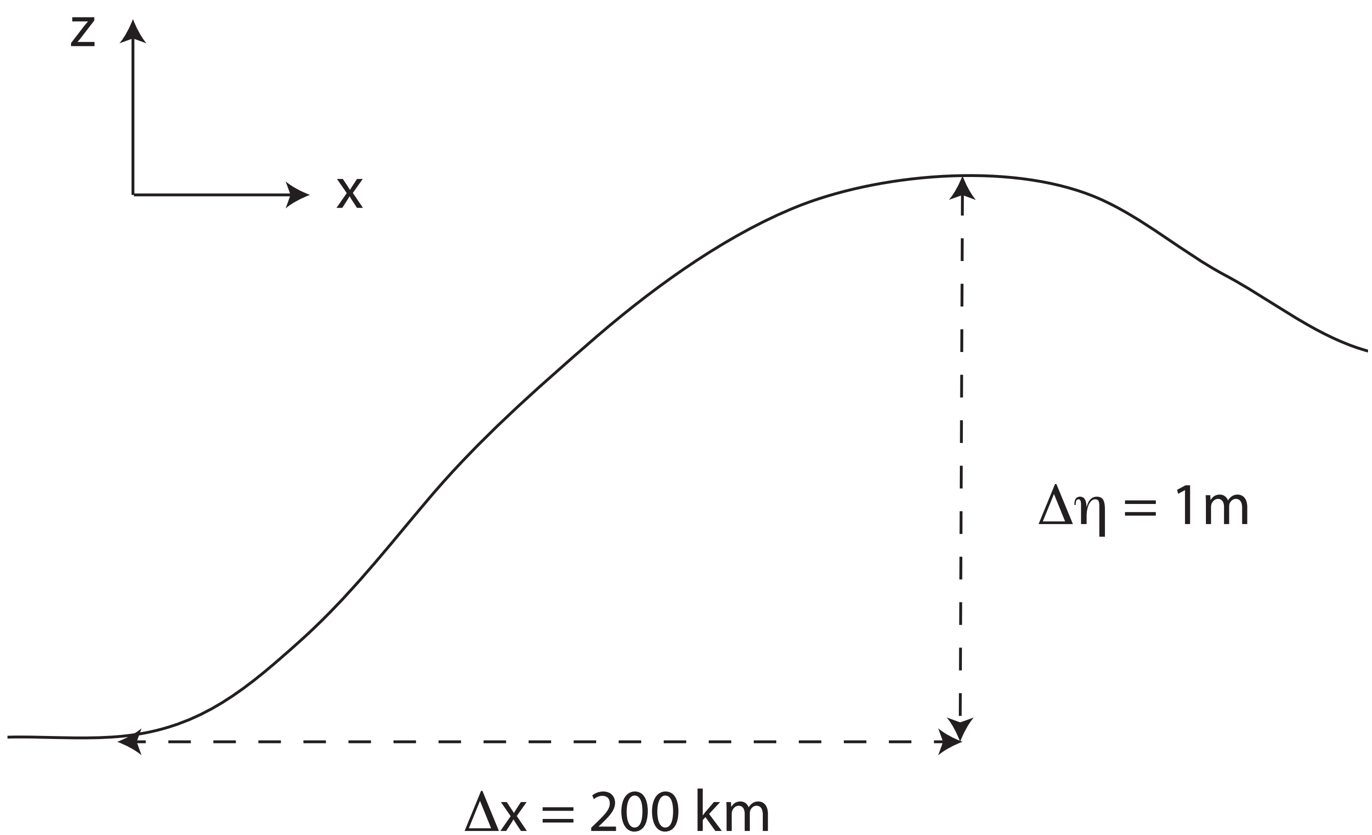
Given the spatial pattern of SSH field, we can calculate the geostrophic flow in the ocean

Satellite SSH in the North Atlantic



Calculating geostrophic current

What is the direction and speed of the geostrophic current with this SSH gradient at 30N (consider Gulf Stream) ?

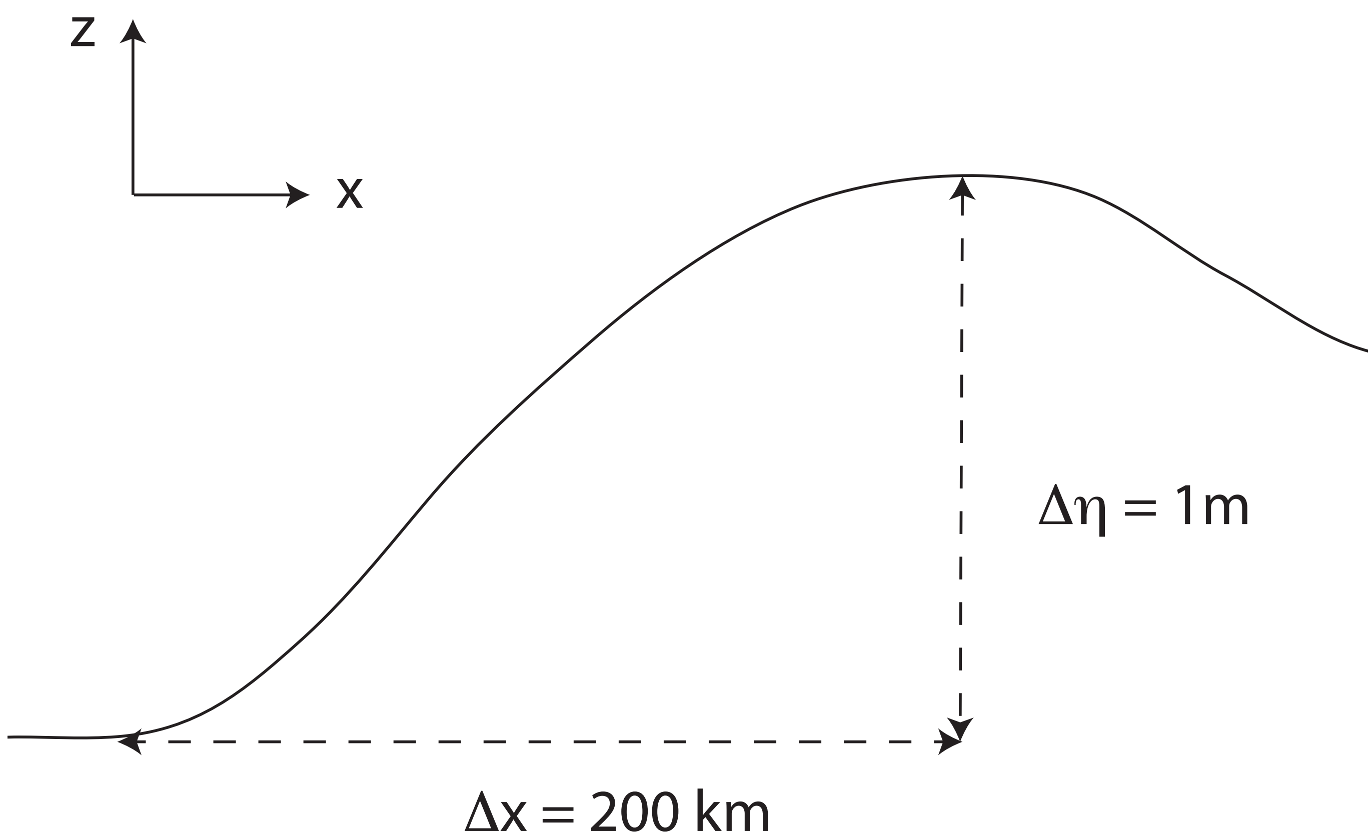


Calculating geostrophic current

What is the direction and speed of the geostrophic current with this SSH gradient at 30N?

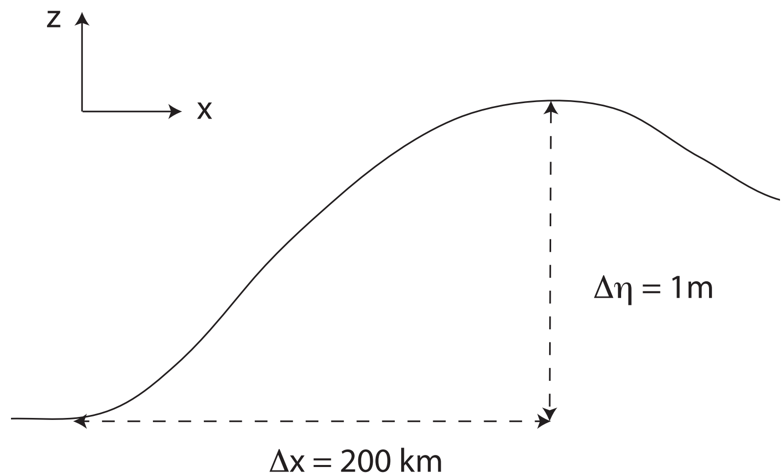
$$-fv_g = -g\frac{\partial \eta}{\partial x}$$

$$v_g = \frac{g}{f}\frac{\Delta \eta}{\Delta x}$$



Calculating geostrophic current

What is the direction and speed of the geostrophic current with this SSH gradient at 30N?



$$-fv_g = -g\frac{\partial\eta}{\partial x}$$

$$v_g = \frac{g}{f}\frac{\Delta\eta}{\Delta x}$$

$$f = 2\Omega\sin\theta = 2\times\frac{2\pi}{86400(\mathrm{s})}\times\sin(30^0)\approx 0.7\times10^{-4}(\mathrm{s}^{-1})$$

$$v_g = \frac{g}{f}\frac{\Delta\eta}{\Delta x} = \frac{9.8(\mathrm{ms}^{-2})}{0.7\times10^{-4}(\mathrm{s}^{-1})}\times\frac{1(\mathrm{m})}{2\times10^{5}(\mathrm{m})} = 0.7(\mathrm{ms}^{-1})$$

Rossby number

- Conditions for geostrophic balance
- Rossby number:

$$\frac{1}{fT}, \frac{U}{fL} \ll 1$$

From the Gulf Stream example:
U = 0.7 (m/s)
f = 0.7 x 10-4 (1/s)
L = 200km = 2 x 10^5 (m)

$$R_T = \frac{1}{fT}$$

$$R_O = \frac{U}{fL} = \frac{0.7(m/s)}{0.7 \times 10^{-4}(1/s) \cdot 2 \times 10^5(m)} = 0.05$$

Cyclonic and anti-cyclonic

Cyclonic = Rotating in the direction of the planetary rotation (CCW in the NH and CW in the SH)

Anti-Cyclonic = Opposite of cyclonic

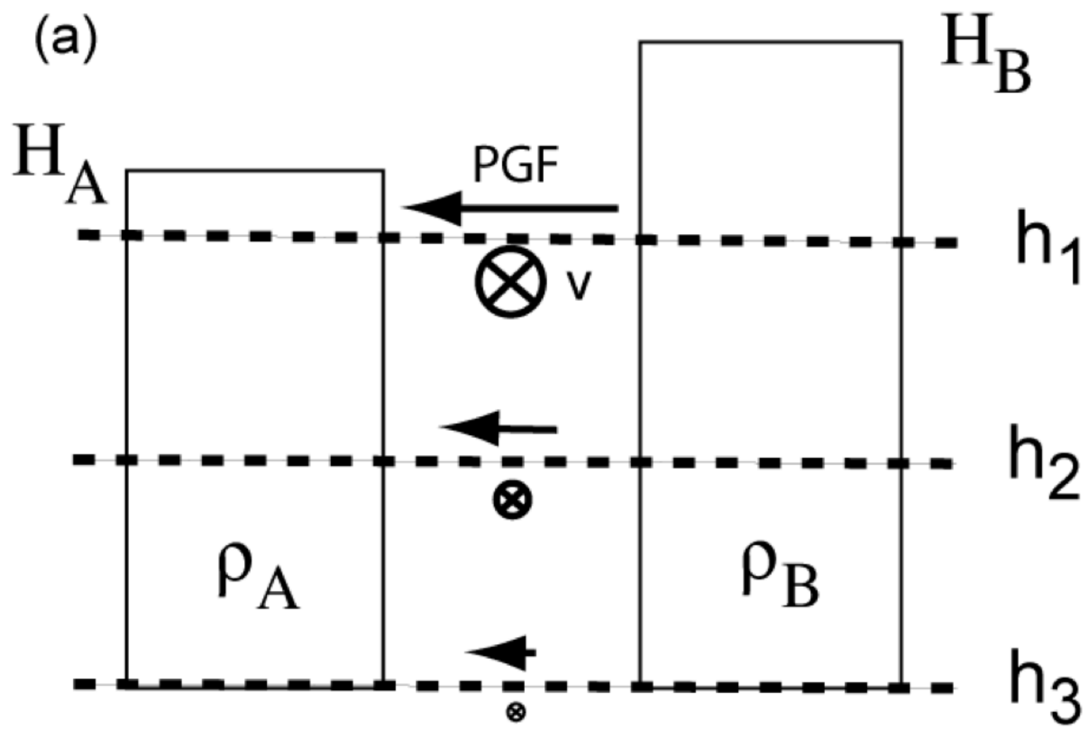
Then, geostrophic flow around low pressure is cyclonic in both northern and southern hemispheres.

Geostrophic flow around low pressure = Cyclonic

Geostrophic flow around high pressure = Anti-Cyclonic

Density and pressure gradient

- Water column with two different densities
- $\rho_A > \rho_B$
- At the surface the less dense water has higher sea surface height
  → Steric sea level
- The pressure gradient gets smaller with depth



Thermal wind balance

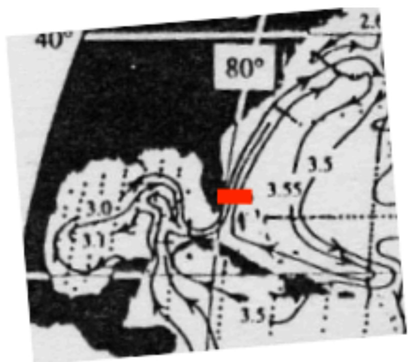
- Assume geostrophy in horizontal, and hydrostatic balance in vertical
- Eliminate pressure

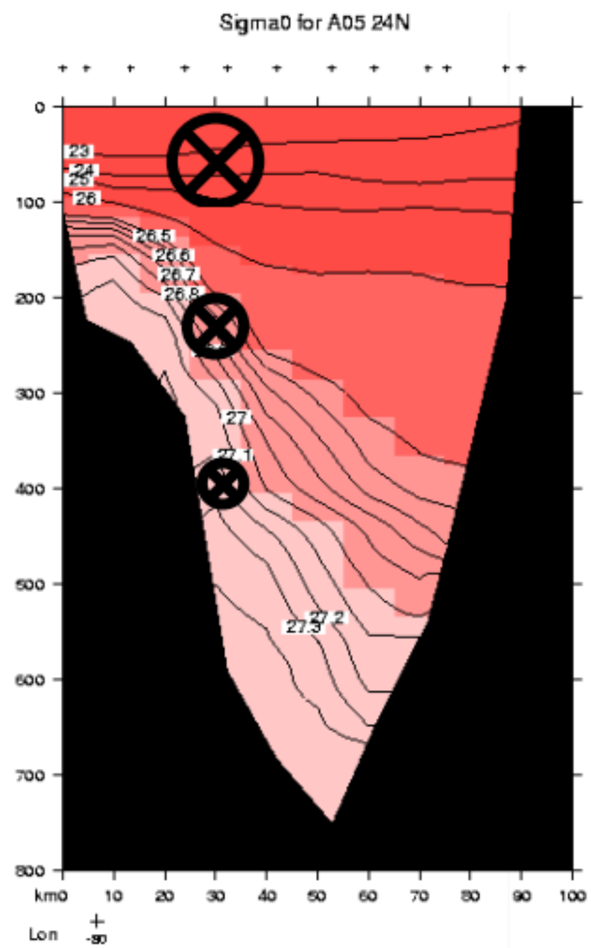
$$\frac{\partial u_g}{\partial z} = \frac{g}{\rho f} \frac{\partial \rho}{\partial y},$$

$$\frac{\partial v_g}{\partial z} = -\frac{g}{\rho f} \frac{\partial \rho}{\partial x}.$$

If we measure $\rho(x,y,z)$ and the value of u and v at a given depth, we can calculate the geostrophic velocity

Observed density profile across the Florida Strait





Thermal wind balance and geostrophic currents

The PGF is calculated as the difference of pressure between two stations at a given depth (relative to the geoid).

a) If the velocity is known at a given depth, then the PGF at that depth is also known (from geostrophy).

b) From the measured density profiles at the two stations, we can calculate how the velocity change with depth.

c) Using the known velocity from (a), which we call the reference velocity, and knowing how velocity changes with depth from (b), we can compute velocity at every depth.

c-alt) ***If a reference velocity is NOT available***, assume deep water is not moving, approximating the reference velocity to be 0 (at arbitrarily set depth in the abyss = level of no motion).

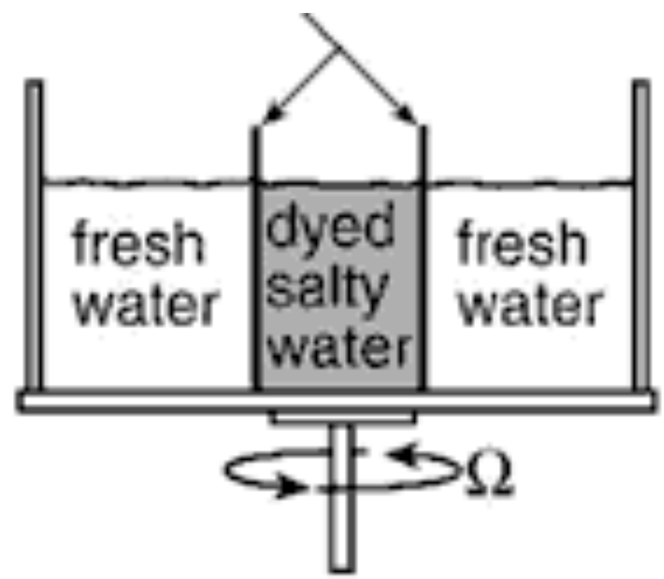
A tank experiment

Objective: illustrate the flow under a density gradient and rapid rotation

Initially, metal container separates dyed salty (dense) water from the surrounding freshwater (less dense)

After the tank is in solid body rotation, the container is removed.

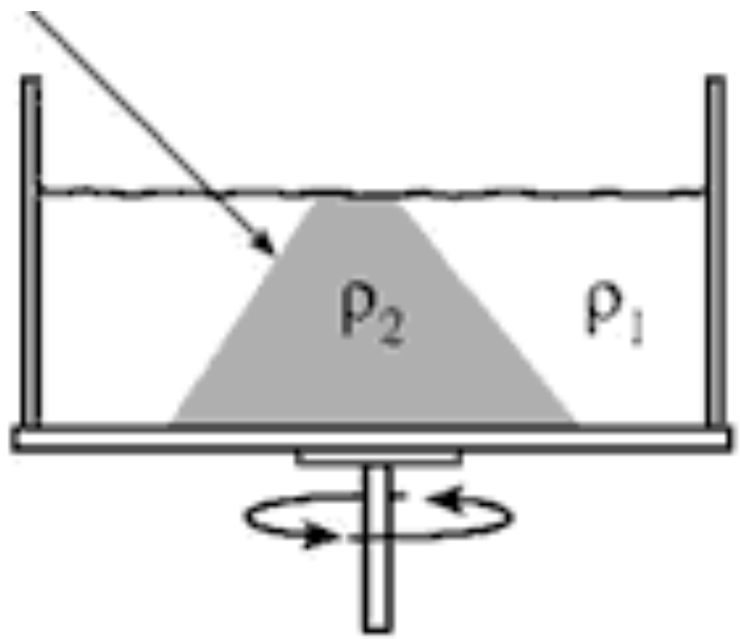
What will happen?



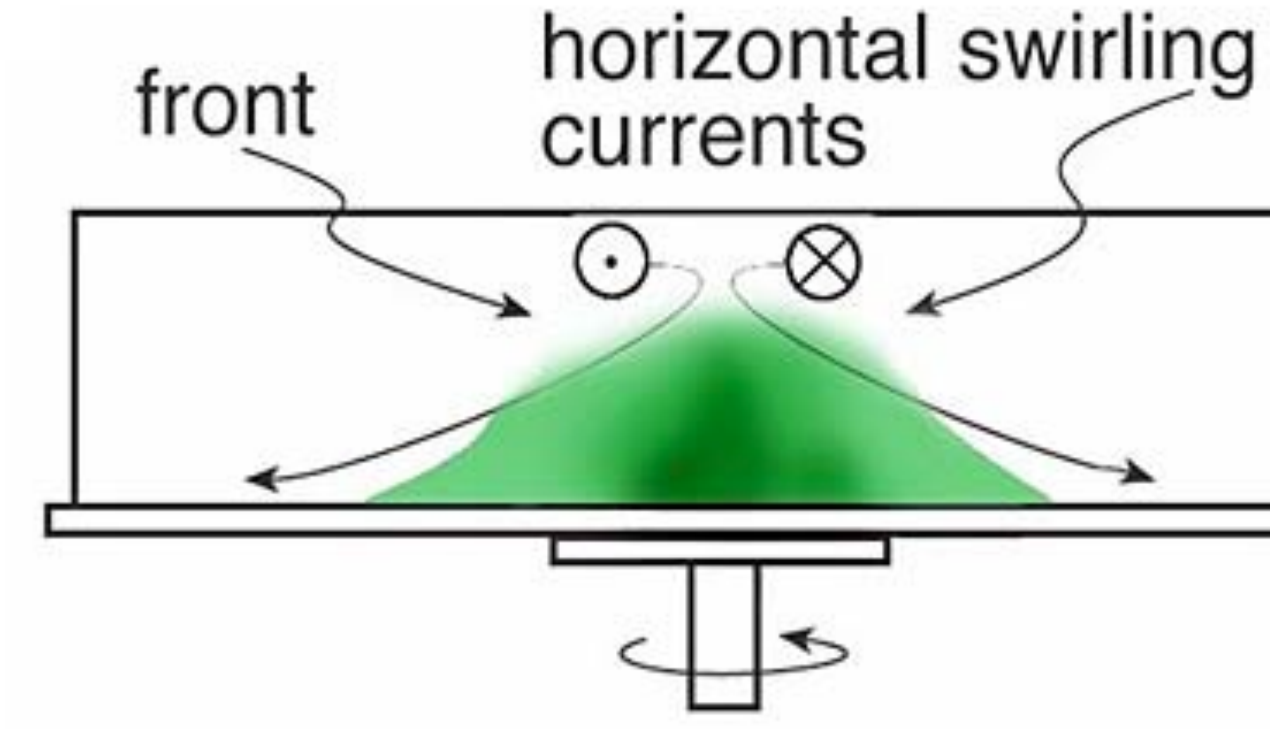
Evolution of circulation and isopycnal tilt

1. Gravity pull down the dense water downward
   Convergence at the top, divergence at the bottom

2. Coriolis effect defects the horizontal motion, the circulation starts to spin around the dense water

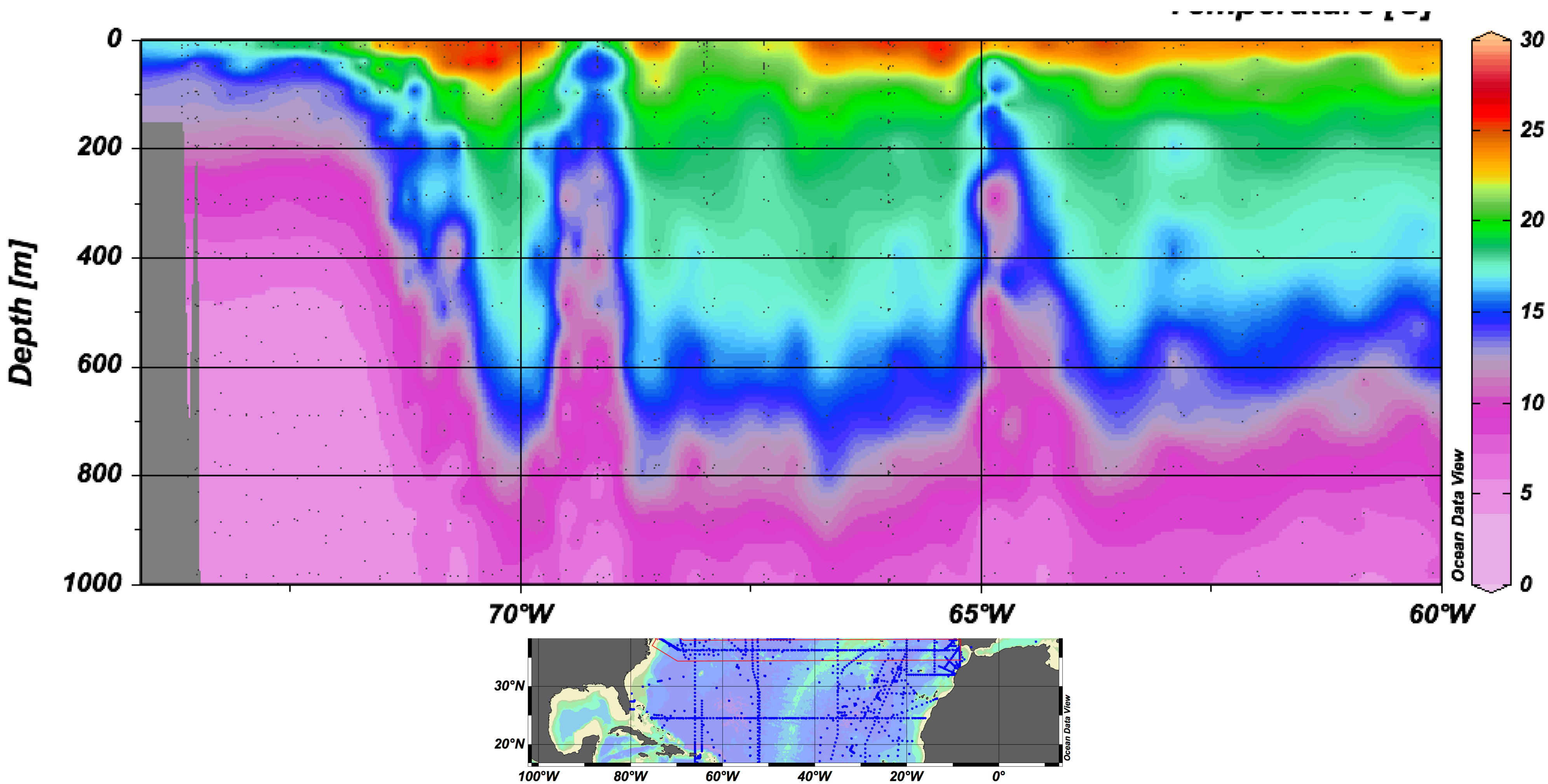


Isopycnal tilt and geostrophic circulation



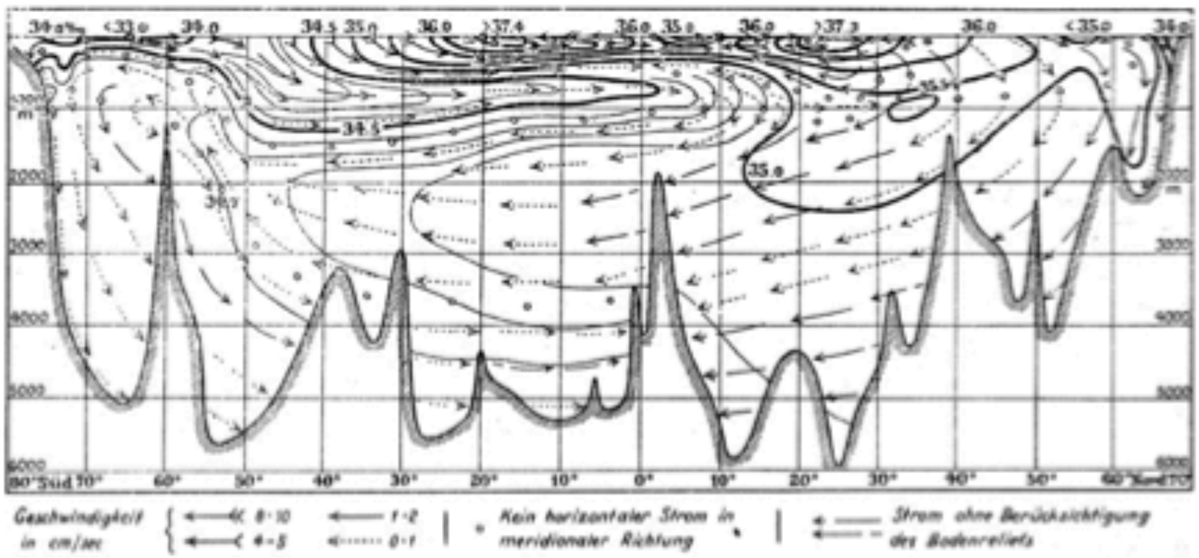
"**Thermal wind balance**" = Coriolis force balancing the buoyancy force acting on the tilted isopycnal surface

Observed isopycnal tilt in the Gulf Stream



Meteor expedition (1925-27)

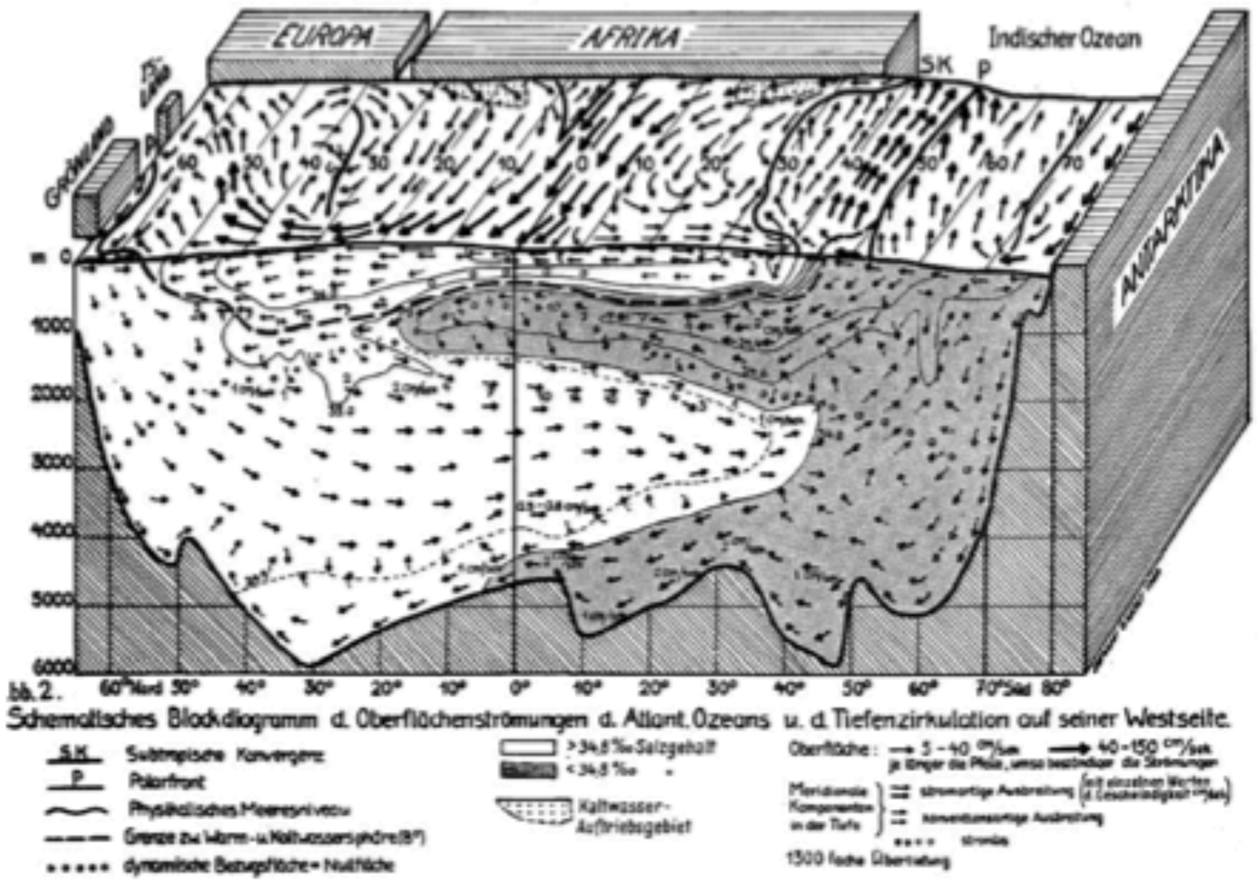
- Testing the thermal wind balance
- Alfred Merz



Merz, 1925

Three dimensional Atlantic circulation after Meteor expedition

G. Wust, 1935; 1949



Summary: week 4

- Geostrophic balance
  - Balance between Coriolis force and Pressure Gradient Force
  - Assume: frictionless and small Rossby number (slow speed, large-scale)
  - YOU WANT: Horizontal current speed and direction
  - YOU NEED: Horizontal pressure gradient at constant depth
  - AVAILABLE MEASUREMENT:
    - Density (T and S)
    - Satellite SSH (after 1992)

Summary: week 4

- Geostrophic balance
  - Cyclonic circulation (CCW in NH) around the low pressure
  - Anticyclonic circulation (CW in NH) around the high pressure
  - Ocean heat content → Steric sea level
    - SSH is higher for the warmer water column
      → Anti-cyclonic flow in subtropics
    - SSH is lower for the colder water column
      → Cyclonic flow in subpolar region

Summary: week 4

- Estimating large-scale ocean currents
  - AVAILABLE MEASUREMENT:
    - Density (T and S)
  - Thermal wind balance + level of no motion
    - Assume that the deep ocean is motionless ($u=v=0$ in the deep ocean, say $z=2{,}000$m), vertically integrate the thermal wind ($\partial u/\partial z$ and $\partial v/\partial z$) vertically.
  - Thermal wind balance + level of known motion (preferred)
    - Obtain velocity at one depth from independent measurement, and then vertically integrate the thermal wind from the known velocity.