

Chapter 2

Differential equations

In the second part of the semester, we develop skills in building models relevant to the Earth, Atmospheric, Oceanic and Planetary Sciences. In particular, differential equations will be used to design and describe such model, and we will also use computers to simulate and visualize such models.

2.1 Geochemical box model

First we consider 'box model' where the box represent a reservoir of chemical elements, such as a layer of soil in the ground, atmospheric air mass or an ocean basin.

Let's consider an example using the Earth's atmosphere as a box. This is a large reservoir, containing

approximately 1.8×10^{20} mol of air molecule. About 80% of the air molecule is nitrogen, and 20% is oxygen. It also contains minor species such as argon and carbon dioxide at a lower concentration. About 0.04% of air molecule is CO_2 . While its concentration is much lower than nitrogen or oxygen, it is an important green house gas and its concentration is changing.

Let's consider a box model of atmospheric CO_2 . Figure 2.1 shows a schematic diagram of the land-atmosphere carbon cycle. The atmosphere contains X mol of carbon, and it varies as a function of time, so we may write $X(t)$. The rate of change of CO_2 is controlled by the input and output of CO_2 for the atmospheric box.

$$\frac{dX}{dt} = (\text{input}) - (\text{output}). \quad (2.1)$$

The input of CO_2 includes respiration (metabolism of living organisms including microbes, plants and animals) and the burning of fossil fuel by humans. The output of CO_2 is driven by the photosynthesis by the land plants. Over the course of one year, the photosynthesis and respiration are balancing one another. Let's approximate the photosynthesis (output) and respiration (input) together as a cosine function of time. The equation 2.1 becomes,

$$\frac{dX}{dt} = B\cos(2\pi t) + F - O \quad (2.2)$$

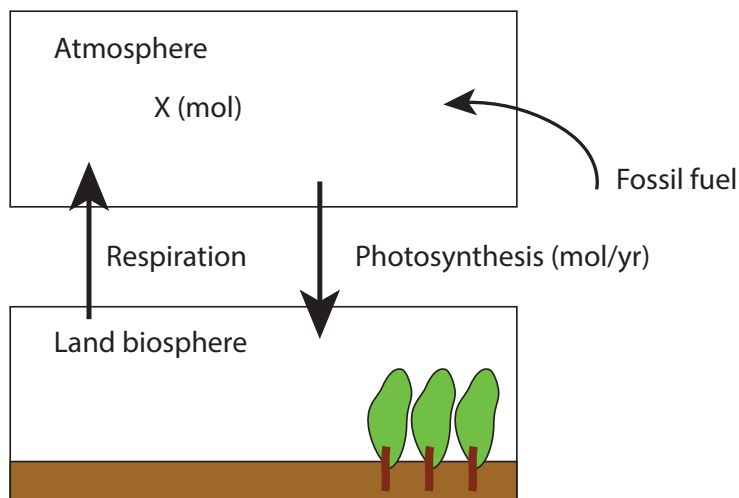


Figure 2.1: A box model for atmospheric CO_2 .

where F is the rate of fossil fuel burning (mol/yr), O is the ocean carbon uptake, and B is the amplitude of the photosynthesis and respiration. For simplicity, let's assume B , F and O are constants.

Let's solve this equation for $X(t)$. We assume there is an initial condition for X , so $X(t = t_0) = X_0$.

$$X(t) = X_0 + \frac{B}{2\pi} \sin(2\pi t) + (F - O)(t - t_0) \quad (2.3)$$

We scale units so X can take the value of ppm in CO_2 . Then, the magnitude of B and F , O are 25, 4

and 2 ppm/yr respectively. Here is an example for the MATLAB script to perform a year-long simulation.

```
% set up model parameters
B=25;
F=4;
O=2;
X0=355;
t0=1991;
% calculate the solution
time=t0+[0:0.01:1];
X=B/(2*pi)*sin(2*pi*time)+(F-O)*(time-t0);
% plot the results
figure(1);
plot(time,X);
xlabel('time');
ylabel('CO2 level');
```

Figure 2.2 shows the result of the box model calculation (eq 2.3). Despite the simplicity of the model, we get approximately correct behavior of the atmospheric CO₂.

There is an alternative way of calculating the evolution of X. Let's look at the equation 2.2 again. We can interpret this equation with a sequence of X with small increment in time,

$$\frac{X_{n+1} - X_n}{\Delta t} = B \cos(2\pi t_n) + F - O$$

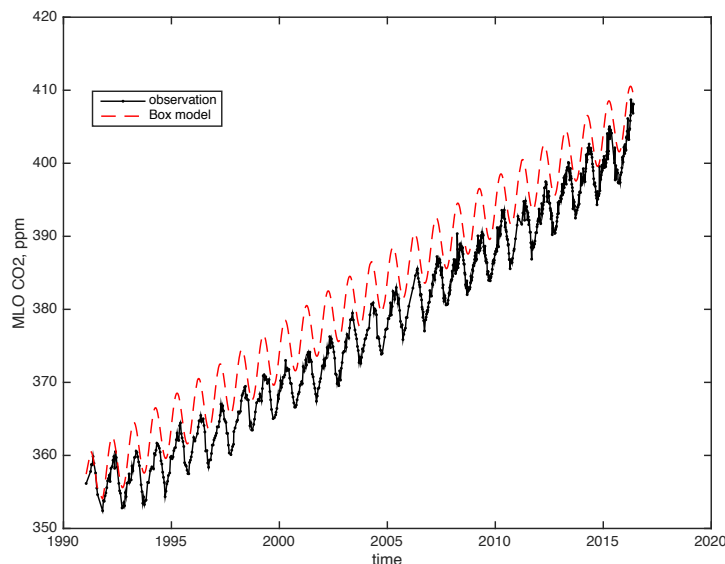


Figure 2.2: Atmospheric CO₂ level observed at the Mauna Loa observatory in Hawaii

$$X_{n+1} = X_n + \Delta t(B\cos(2\pi t_n) + F - O) \quad (2.4)$$

which means, given an initial condition in X , the equation 2.4 can be used to predict the value of X for the next time step. This is called time stepping. Given a value for X_n , you can calculate the next time step X_{n+1} . Then, using the value for X_{n+1} , you can get X_{n+2} . We can repeat the procedure to march forward in time.

In practice, this approach only works when you take

a small time step (Δt) because we assume that the gradient does not change over the small time step. A larger time step will cause error in your solution. Here is an example for the MATLAB script to perform 100 timesteps covering 1 year.

```
% setting up the model parameters
N=100;
dt=0.01;
B=25;
F=4;
O=2;
% setting up initial condition
X(1)=355;
time(1)=1991;
% time stepping loop
for n=1:N-1
    time(n+1)=time(n)+dt;
    X(n+1)=X(n)+dt*( B*cos(2*pi*time(n)) + (F-O) );
end
% plot the result
figure(1);
plot(time,X);
xlabel('time');
ylabel('CO2 level');
```

Exercises We will examine the analytic and numer-

ical solution to the box model equation 2.2.

1. Reproduce Figure 2.2, comparing the observed CO₂ and analytic solution.
2. Overlay numerical solution with the time step of 0.01 year.
3. Modify the model where the emission is increasing with time, $F(t) = F_0 + F_1(t-t_0)$, where $F_0 = 4$ PgC/yr and $F_1 = 0.4$ PgC/yr². t_0 is the initial time (1991). Calculate the analytic and numerical solutions.
4. Publish your MATLAB script that performs above tasks.