2.6 Diffusion equation

The last major component of this class is so-called diffusion (or heat) equation. It is motivated by the transfer of heat across medium. The heat tends to spread from the hot region to the cool region. The physics behind it is the heat conduction by the Brownian motion, leading to the down-gradient diffusion.

The flow of heat in x-direction can be expressed as

$$F_x = -k\frac{\partial T}{\partial x} \tag{2.46}$$

where T is the temperature and k is the diffusivity coefficient in units of m^2s^{-1}. The negative sign indicates that it is a down-gradient flux as the heat flows from high to low temperature regions.

When there is a convergence of heat, we anticipate a warming in the medium. Thus the rate of heating can be written as

$$\begin{aligned} \frac{\partial T}{\partial t} &= -\frac{\partial F_x}{\partial x} \\ &= k\frac{\partial^2 T}{\partial x^2}. \end{aligned} \tag{2.47}$$

The Eq 2.47 is called the "heat equation" or diffusion equation. It predicts that the rate of local heating is proportional to the curvature (second derivative) of the temperature profile. If there is a local peak in the temperature profile, it will cool down because the heat escapes from it. The rate of heating/cooling is dictated by the diffusivity coefficient (k).

Analytic solution

In some cases it is possible to derive analytic solution. Consider a metal bar of length L. Both ends of the bar is set to a constant temperature ($T = 0$). The bar is heated to a constant temperature ($T = T_0 > 0$) at $t = 0$. What would be the evolution of temperature $T(x, t)$ in the bar?

We solve this by the separation of variable. Assume a solution exists in the form of $T(x, t) = X(x)T(t)$. Substitute this into the Eq 2.47. After a few manipulations, we find a set of ODEs

$$\frac{dT}{dt} = -\lambda k T \tag{2.48}$$

$$\frac{d^2X}{dx^2} = -\lambda X \tag{2.49}$$

The first equation (2.48) predicts an exponentially decaying solution in time, and the second (2.49) predicts a set of sinusoidal (sin and cos) solution in space.

$$T(t) = e^{-\lambda k t} \tag{2.50}$$

But the value of λ is yet to be determined.

The second equation (2.49) has a solution of the form

$$X(x) = Asin(\sqrt{\lambda}x) \tag{2.51}$$

where A is a constant. Because of the boundary condition ($X = 0$ at $x = 0, L$), the value of λ is set such that

$$\sqrt{\lambda}L = n\pi \qquad n = 1, 2, 3... \tag{2.52}$$

The solution can be expressed as the superposition of many sine functions. The exact combination is determined by the initial condition. At $t = 0$, we have $T = 1$ and the sum of the Eq 2.50 must equal T_0.

$$T_0 = \Sigma \left\{ T_n sin\left(\frac{n\pi x}{L}\right) \right\}. \tag{2.53}$$

This is a Fourier sine series, and the value of T_n can be calculated as

$$T_n = \frac{4T_0}{n\pi} \qquad n = 1, 3, 5... \tag{2.54}$$

Fig 2.9 shows how the sine waves can represent the uniform constant through superposition.

So the full solution is

$$T(x, t) = \Sigma T_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} sin\left(\frac{n\pi x}{L}\right) \tag{2.55}$$

Examining the form of the solution, we see that higher wave number (n) decays faster, leaving behind the low

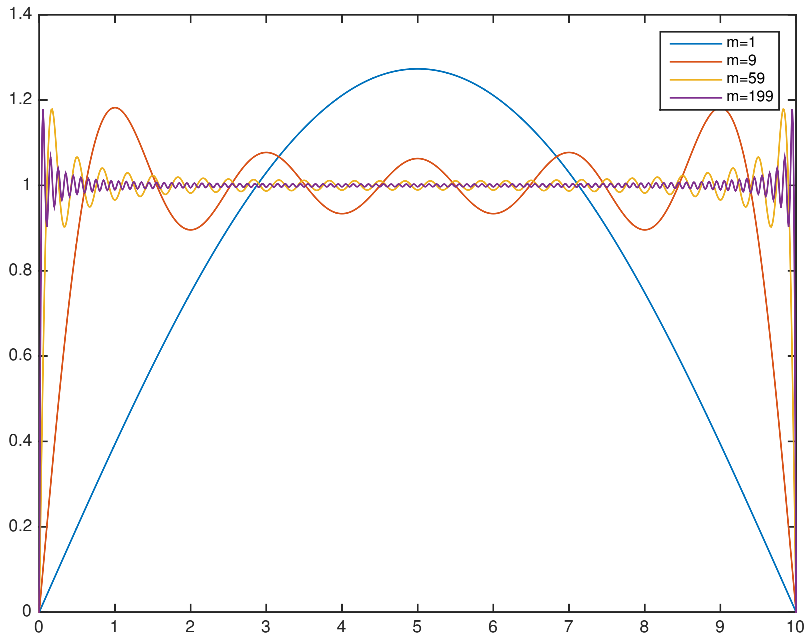

Figure 2.9: Partial sum of the Fourier sine series for the constant initial condition.

wave number modes. The analytic solution can be plotted as a function of time. For the case of $L = 10m$ and $k = 10^{-2}m^2s^{-1}$, Fig 2.10 shows the solution at t = 1, 10, 100, and 1,000 seconds. It starts off cooling at the edges and the temperature gradually decreases towards the middle of the domain.

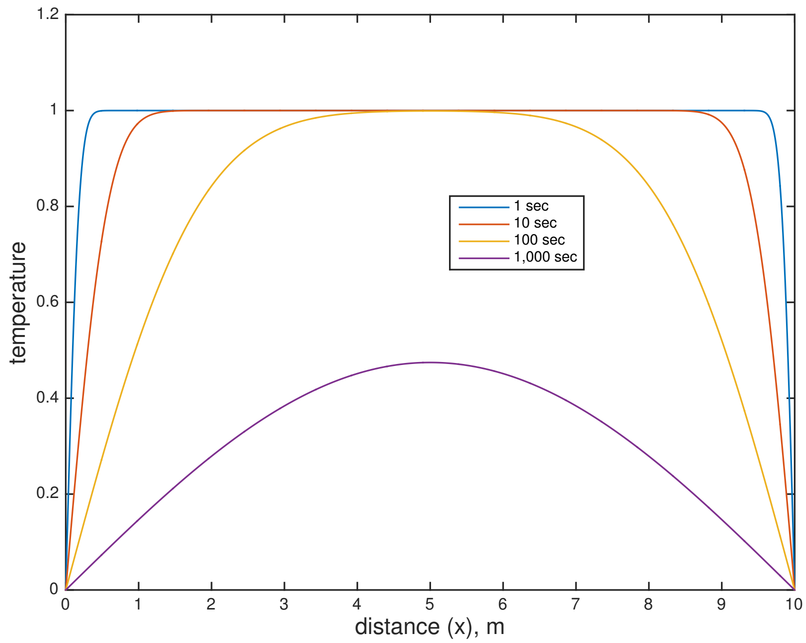

Figure 2.10: Analytic solution to the heat equation at time t=1, 10, 100 and 1,000 seconds.

2.7 Numerical solution to the heat equation

Now that we solve the heat equation numerically. Similar to the wave equation, we take the grid point approach where the domain is divided into small chunks of Δx.

Because the boundary condition for temperature is fixed at $x = 0, L$, it is good to set the temperature grid on the boundaries. Then the heat flux is defined in between the temperature grids.

$$F_n = -k\frac{T(n) - T(n-1)}{\Delta x} \tag{2.56}$$

At the nth temperature grid point, the incoming heat flux is F_n and the outgoing heat flux is F_{n+1}. The net temperature increase is

$$\begin{aligned} \frac{\partial T}{\partial t} &= \frac{F_n - F_{n+1}}{\Delta x} \\ &= k\frac{T(n+1) - 2T(n) + T(n-1)}{\Delta x^2} \end{aligned} \tag{2.57}$$

Taking the Euler forward method in the time stepping, we get

$$\begin{aligned} T(n, t+\Delta t) &= T(n,t) - \frac{\Delta t k}{\Delta x^2} \\ &\quad \{T(n+1,t) - 2T(n,t) + T(n-1,t)\} \end{aligned} \tag{2.58}$$

A numerical scheme is implemented with the grid spacing of $\Delta x = 0.2$m and the time step of $\Delta t = 1$ sec. The the Courant number for this problem is $\frac{k\Delta t}{\Delta x^2}$. For this parameter choice, the value of C is 0.25 so it satisfies the Courant condition.

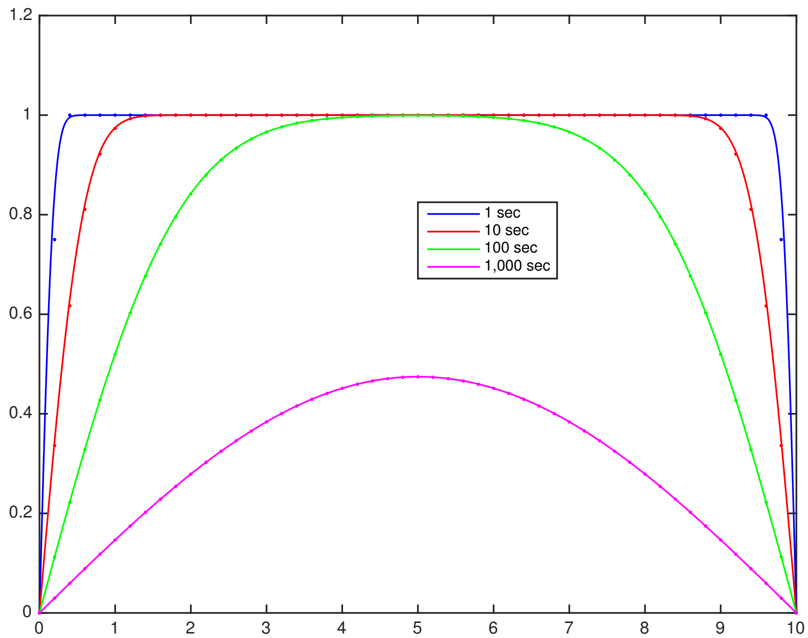

Figure 2.11: Comparing the numerical and analytic solutions to the heat equation.