2.3 Partial differential equation

In the previous section, our model started to resolve both spatial and temporal variations as a system of coupled ODEs. The global ocean was divided into 3 boxes, resulting in a coupled three ordinary differential equations (ODEs). If we want to better resolve the spatial pattern of phosphorus, we can continue to add more and more boxes representing the global ocean. It will lead to many boxes and differential equations to represent them.

Partial differential equations (PDEs) provide a mathematical framework to treat variations in more than one dimensions (such as time and space), and is often used in the Earth, atmospheric, oceanic and planetary sciences.

Consider waves. Waves propagates information (velocity, pressure, etc) across time and space. In order to model wave propagation, we have to model the physical system not only how it evolves over time (like we did for the box models) but also how it propagates in space. This requires the use of PDEs.

We start off our discussion of PDEs using wave equations as an example. We first analyze the property of waves and the class of PDEs called wave equation. We then develop a numerical approach to solve for the wave equation using MATLAB.

Plane wave

An infinite plane wave, in a simplest form, can be described using a sine and cosine function.

$$f(x,t) = Asin(kx - \omega t) \tag{2.31}$$

where k is the wave number and ω is the angular frequency. As the name indicate, k measures how many waves you can fit within the length of 2π. The wave length (λ) can be calculated as the distance in x where the wave makes a full cycle, $k\lambda = 2\pi$. In a similar way, angular frequency (ω) measures how many cycles of wave can occur within the time of 2π. The wave period (T) can be calculated as the time in takes for the wave to make a full cycle, $\omega T = 2\pi$.

The ratio between ω and k determines the wave speed, $c = \frac{\omega}{k}$ as the phase angle of the wave remains constant if $x = ct$.

Superposition of two plane waves

Now consider the superposition of two plane waves with slightly different wave number and angular frequency.

$$f_A = Asin\left\{(k - \delta k)x - (\omega - \delta\omega)t\right\} \tag{2.32}$$

$$f_B = Asin\left\{(k + \delta k)x - (\omega + \delta\omega)t\right\} \tag{2.33}$$

where δk and $\delta\omega$ are small changes and their magnitudes are much smaller than k and ω respectively. With a few manipulations, we can show that the combined wave has the form

$$f_A + f_B = 2A \underbrace{sin(kx - \omega t)}_{fast,short} \underbrace{cos(\delta kx - \delta\omega t)}_{slow,long} \qquad (2.34)$$

Thus the interference pattern that results from the superposition of the two similar waves has the slowly changing, long wave component. This occurs when you hear beats tuning the guitar strings. When the strings are not perfectly tuned, they give slow beats and you would adjust the tension of the string until the beats disappear.

Phase and group velocity

Here there are two types of wave propagation. As before, the ratio between ω and k determines the wave speed, $c = \frac{\omega}{k}$. This is called phase velocity, it captures the propagation speed of the phase angle of the wave. There is another velocity associated with the pattern of wave interference. The ratio between $\delta\omega$ and δk determines the speed at which the interference propagates, $c_g = \frac{\delta\omega}{\delta k}$. This is called group velocity. The group velocity determines the propagation of energy in the geophysical waves and plays important roles in meteorology and oceanography. In general the group velocity can be defined as

$$c_g = \frac{\partial\omega}{\partial k}. \qquad (2.35)$$

In general, the dependence of frequency (ω) on wave number (k) is termed as dispersion relation. For some waves, such as sound wave, $c = c_g =$ constant for all frequencies. Regardless of the pitch of our voice, the sounds we make travels through the air at the same speed. This allows us to communicate by speech. For other waves, however, c can vary depending on the frequency. When you throw a stone in a quiet pond, it makes a concentric circle that spreads out. In this case the water wave is dispersive. The longer wave tends to travel faster than the short waves, leaving behind the short waves. Since the dispersion relation can be used to characterize the wave propagation, it often becomes the primary objective of the studies of geophysical waves.

Exercise

1. Consider a tsunami wave with a speed of $\sqrt{gH}$ where H is the depth of the ocean. With the average ocean depth of 4km in the Pacific, how long does it take for it to travel across the Pacific (assuming 10,000km wide)?

2. What is the wave length of the above tsunami wave at the frequency of $\omega = 1(min^{-1})$?

3. Show that the wave function, $h(x,t) = Asin(kx -$

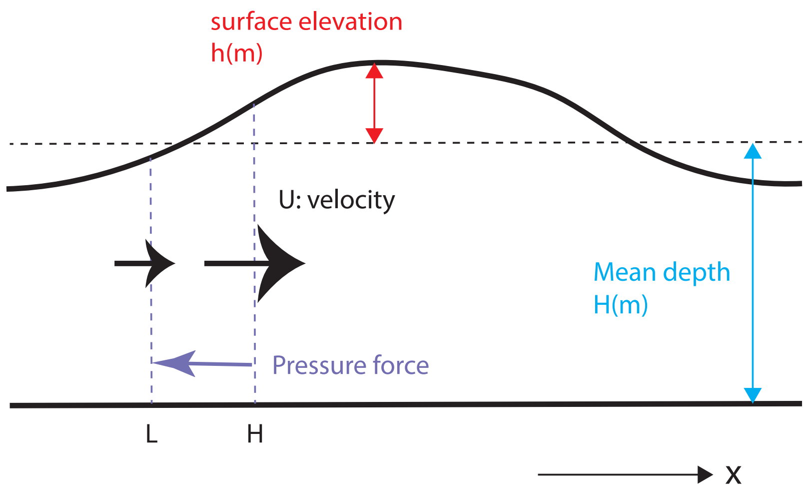

Figure 2.5: A schematic diagram for the shallow water model.

ωt), satisfies the partial differential equation, $\frac{\partial^2 h}{\partial t^2} = c^2 \frac{\partial^2 h}{\partial x^2}$ where c is the wave speed.

2.4 Shallow water equation

Here we examine the one-dimensional shallow water equation. Fig 2.5 schematically describes the model. It consists of a fluid of mean depth of H with one-dimensional velocity $u(x, t)$ (black arrow). The surface of the fluid can be slightly elevated or depressed with anomalous surface height of $h(x, t)$ (red arrow).

Surface height controls the distribution of water pressure. Elevated surface level means there is more pressure underneath. Horizontal variation in the surface height generates pressure variation that accelerates the water. The Newton's law in x-direction can be written as

$$M\frac{\partial u}{\partial t} = -Mg\frac{\partial h}{\partial x} \tag{2.36}$$

where M is the mass of the water per unit area (density × mean depth) and is assumed to be a constant. The negative sign on the RHS indicates that the pressure force acts from the high to low pressure region.

When there is a horizontal velocity variations, there can be accumulation of mass locally. The statement of mass conservation can be expressed as

$$\frac{\partial h}{\partial t} = -H\frac{\partial u}{\partial x}. \tag{2.37}$$

Combining the equation of motion (Newton's law) and the conservation of mass gives us the wave equation. If the velocity u is eliminated between the two Eqs (2.36,2.37), we get

$$\frac{\partial^2 h}{\partial t^2} = c^2\frac{\partial^2 h}{\partial x^2} \tag{2.38}$$

where c is the wave speed defined as $\sqrt{gH}$. This form of partial differential equation is commonly called the "wave equation".

Apply the form of $h(x,t)$ in the plane wave (Eq 2.31) and find the dispersion relation as

$$\omega = \pm ck. \tag{2.39}$$

This means that there are two classes of solutions to this wave equation. One is propagating in the positive x direction at the speed of c, and another solution propagates in the negative x direction at the same speed.

There could be many solutions to the above wave equation. The dispersion relation only determines the wave propagation. There are infinite possibilities for its particular phase and amplitudes, so actual solutions depend on the initial and boundary conditions.

2.5 Numerical solutions to the shallow water equation

Consider the shallow water wave in a well with the horizontal extent of $-L \leq x \leq +L$. At the edges there is no flow into and out of the well ($u = 0; x = \pm L$). This section develops a numerical algorithm to solve the wave equation in this domain.

Here, we take a simple, regular grid approach where we divide the domain into small chunks of equal size in Δx, and we calculate the fluid velocity (u) and water height (h) for each of the grid points. Fig 2.6 shows the

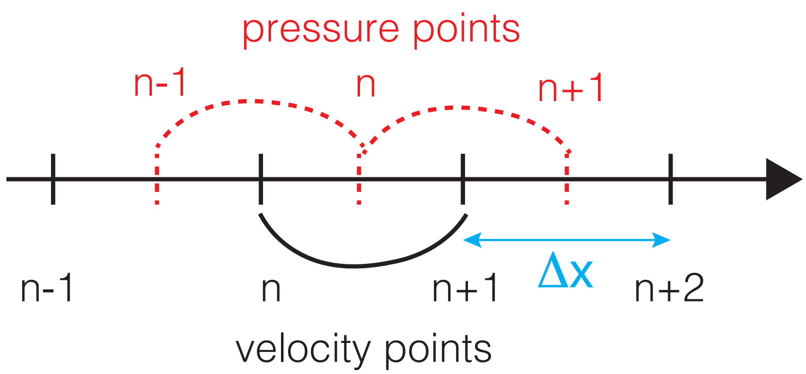

Figure 2.6: A schematic diagram for the staggered grid.

definition of the grid points. We also "stagger" the velocity and pressure (h) grid points where they are defined with a offset of half Δx. The reason for setting up staggered grid becomes evident below. It also makes sense to place the velocity points overlapping the domain boundary at $x = \pm L$, so it is easy to implement the boundary condition.

We choose to use the Eqs 2.36 and 2.37. Let's first consider the Newton's law (Eq. 2.36). This equation involves the evaluation of the pressure gradient to determine the acceleration of the velocity. To calculate the acceleration at the velocity point (n), we need to evaluate the surface height gradient at that point.

$$\left[\frac{\partial h}{\partial x}\right]_n \sim \frac{h(n,t) - h(n-1,t)}{\Delta x}. \tag{2.40}$$

Using this representation of the height gradient, we can consider performing the Euler forward time stepping for the velocity u.

$$u(n, t + \Delta t) = u(n, t) - \Delta t g \left[\frac{\partial h}{\partial x}\right]_n . \tag{2.41}$$

Similarly, we can approximate the gradient of velocity at the pressure (height) points to integrate the mass conservation equation.

$$\left[\frac{\partial u}{\partial x}\right]_n \sim \frac{u(n+1, t) - u(n, t)}{\Delta x}. \tag{2.42}$$

$$h(n, t + \Delta t) = h(n, t) - \Delta t H \left[\frac{\partial u}{\partial x}\right]_n . \tag{2.43}$$

Given an initial condition in u and h, we can step forward the model to simulate the wave equation. As an example, a simulation with the mean depth of $H = 1m$ and $L = 5m$ is shown in Fig 2.7. The grid is set up with a $0.2m$ grid spacing and the time step is set to $0.01s$. The initial condition is $u = 0$ everywhere and $h = 0.2e^{-(x+L)}$. This creates a wave that initially propagate in the positive x direction, and then it bounced against the wall.

Despite its simplicity, the model can simulate the wave propagation reasonably well. Approximately it took 3 seconds for the wave to travel across the well width of 10m. This implies the wave speed of about $3ms^{-1}$. Based

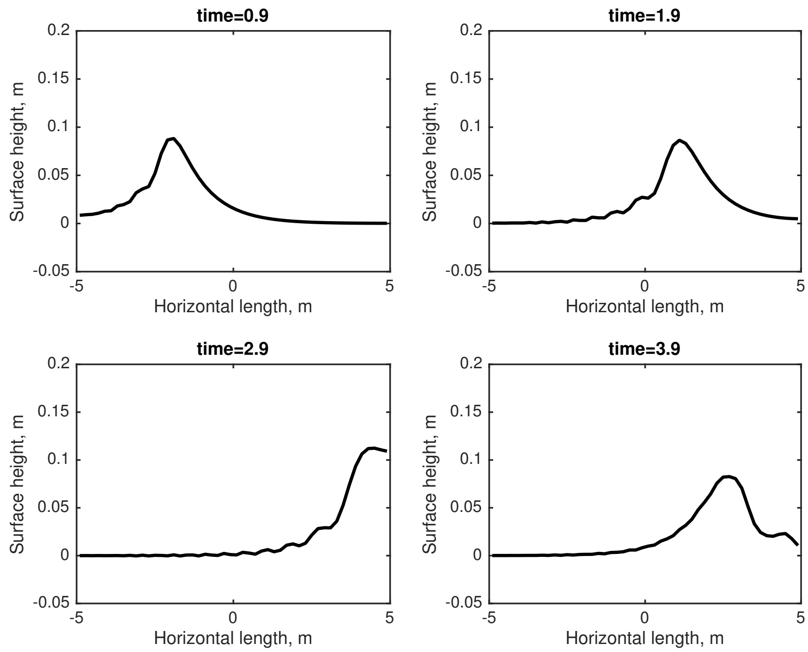

Figure 2.7: Wave simulation with the Euler forward scheme. These are snapshots at 1 second intervals.

on the depth of $1m$, we get the wave speed $c = \sqrt{gH} = 3.1ms^{-1}$. In this application, however, the Euler forward scheme develops a significant numerical 'noise' following the wave crest.

Predictor-corrector method

The issue of numerical noise can be addressed by the application of more stable time stepping scheme. Here, we use a simple predictor-corrector method. It aims to achieve the stability of the Euler backward scheme. In this context, the velocity is first stepped forward using the Eular forward scheme. Then, using the velocity gradient at $t+\Delta t$, the surface height is then stepped forward in time. Once the surface height is determined at $t+\Delta t$, then re-calculate the velocity using the updated surface height fields.

$$\begin{aligned} u_p(n, t+\Delta t) &= u(n,t) - \Delta t g \left[\frac{\partial h}{\partial x}\right]_n . \\ h(n, t+\Delta t) &= h(n,t) - \Delta t H \left[\frac{\partial u_p(t+\Delta t)}{\partial x}\right]_n . \\ u(n, t+\Delta t) &= u(n,t) - \Delta t g \left[\frac{\partial h(t+\Delta t)}{\partial x}\right]_n \end{aligned} \tag{2.44}$$

u_p is only used to calculate the gradient of u at $t+\Delta t$ and its value is not retained. The last equation is the 'correction' step which is calculated based on the surface height gradient at $t+\Delta t$.

Stability of numerical schemes

Comparing Figs 2.7 and 2.8, the predictor corrector method produces more realistic, smooth wave profiles. More stable schemes are more tolerant to the numerical noises given a set of time steps and grid spacing. If we take a time step that is too long, any numerical scheme is not

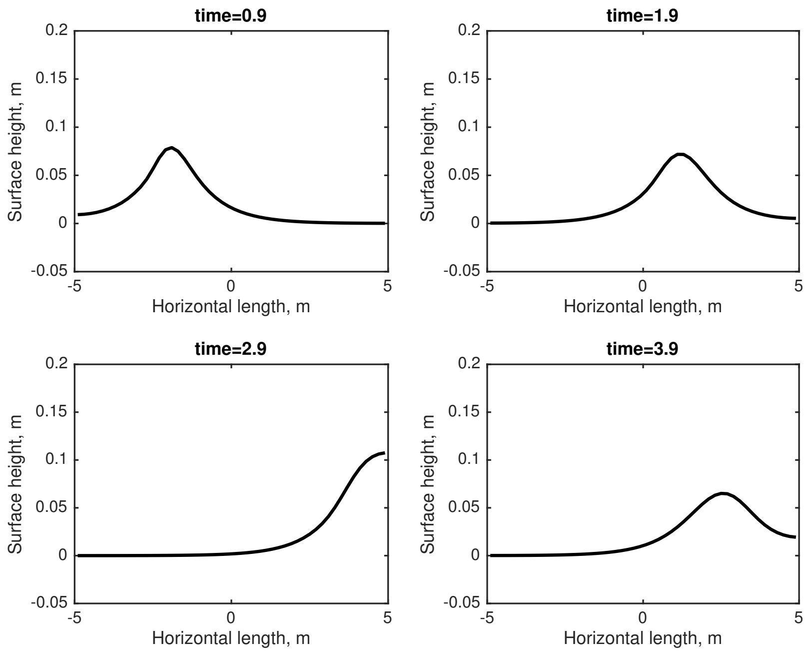

Figure 2.8: Wave simulation with the predictor-corrector scheme. These are snapshots at 1 second intervals.

immune to the instability. There is a principle that regulates the stability of the computational schemes under different sizes of time steps and grid spacing.

The Courant–Friedrichs–Lewy (CFL) condition is a necessary (but not sufficient) condition for convergence of numerical solutions to partial differential equations. The Courant number is a dimensionless number defined as

$$C = \frac{c\Delta t}{\Delta x}. \tag{2.45}$$

where c is the fastest wave speed represented in the model (in our case, $\sqrt{gH}$). The CFL condition states that the Courant number (C) must be smaller than 1. If the value of C is greater than 1, a wave can travel across more than one grid point within a time step. This will cause numerical instability.

In the example above, we used the mean water depth of 1m, we have about $c = 3ms^{-1}$. The grid spacing is $0.2m$. Thus we must use the time steps that are shorter than $0.066s$. We used the Δt of $0.01s$, so in this case we have $C = 0.15$ and it satisfied the CFL condition.

Exercises

1. Reproduce the calculation shown in Fig 2.8. Make sure that the propagation of the wave is consistent with the theoretical value.

2. Simulate the wave propagation for a longer time period. Let's say for 30 seconds. Observe and comment on the amplitude of the wave and how it changes with time.

3. Examine the simulation with an increased time step. Perform several simulations. Observe and comment on the behavior of your solution.

Advanced exercises

1. Generate a two dimensional shallow water model by including the Newton's law in y-direction.